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Linear System of Equations, Vector Linear Combination, Vector Matrix

A Linear System of Equations is a collection of two or more equations involving two or more variables x_p each within a power function raised to the first power in the numerator only but not together within the same term, not within any exponentials, logarithms, trig functions, or inverse trig functions, each multiplied by a scalar a_{pq} , summed together, and related to a fixed constant b_p. A Linear System of Equations can be converted into three other useful forms for algebra analysis, the Vector Linear Combination Form, the Vector Matrix Form, and the Augmented Vector Matrix Form.

Linear Systems of Equations, Vector Linear Combination, and Vector Matrix, and Augmented Vector Matrix

Linear Systems of Equations has variables x_p with coefficients a_{pq} summed together and related to fixed constants b_p.

Linear System of Equations Form

A Linear System of Equations has the general form with variables x_p and coefficients a_{pq} and related to constants b_p:

$$\begin{matrix} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \cdots + a_{1n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \cdots + a_{2n}\,x_n = b_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 + \cdots + a_{3n}\,x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + \cdots + a_{mn}\,x_n = b_m \end{matrix}$$

The set of all combinations of x_p values that make the Linear System of Equations hold true is called the solution set.

Vector Linear Combination Form

A Vector Linear Combination has the general form with variables x_p and coefficient vectors $\overrightarrow{v_p}$ related to vector $\overrightarrow{b_p}$:

$$x_1\,\overrightarrow{v_1} + x_2\,\overrightarrow{v_2} + x_3\,\overrightarrow{v_3} + \cdots + x_n\,\overrightarrow{v_n} = \vec{b} \qquad \mathit{Compact\ Version}$$

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \qquad \mathit{Expanded\ Version}$$

The set of all combinations of x_p values that make the Vector Linear Combination hold true is called either the solution set, the weights of the vectors $\overrightarrow{v_p}$, the scalar multiples of the vectors $\overrightarrow{v_p}$, or the coordinates for the vector $\overrightarrow{b_p}$.

If there exists at least one combination for x_p values that make the Vector Linear Combination hold true, the vector $\overrightarrow{b_p}$ is called dependent on the set of coefficient vectors $\overrightarrow{v_p}$ and the vector $\overrightarrow{b_p}$ is in the span of the set of coefficient vectors $\overrightarrow{v_p}$.

If there exists no combination for x_p values that make the Vector Linear Combination hold true, the vector $\overrightarrow{b_p}$ is called independent on the set of coefficient vectors $\overrightarrow{v_p}$ and the vector $\overrightarrow{b_p}$ is not in the span of the set of coefficient vectors $\overrightarrow{v_p}$.

Vector Matrix Form

A Vector Matrix has the general form with vector variable $\overrightarrow{x_p}$ and vector coefficient matrix A related to vector $\overrightarrow{b_p}$:

$$A\,\vec{x} = \vec{b} \qquad \mathit{Compact\ Version}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \qquad \mathit{Expanded\ Version}$$

The set of all combinations of x_p values that make the Vector Matrix hold true is called the solution set.

Augmented Vector Matrix Form

A Augmented Vector Matrix has the general form vector coefficient matrix A related to vector $\overrightarrow{b_p}$:

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

The set of all combinations of x_p values that make the Vector Matrix hold true is called the solution set.

Matrix and Vector Arithmetic

A Matrix is a rectangular arrangement of numbers that represent coefficients to a linear system of equations with form

$$A_{m\,n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

The Matrix $A_{m\,n}$ has a size m by n that represents the number of rows m by the number of columns n in the matrix.
The Matrix $A_{n\,n}$ has a size n by n with number of rows n by the number of columns n is known as a Square Matrix.
The Matrix $A_{m\,n}$ has m number of rows that each have n number of components within and the rows exist in $\mathbb{R}^n$
The Matrix $A_{m\,n}$ has n number of columns that each have m number of components within and the columns exist in $\mathbb{R}^m$
The Vector x_n is a Matrix with just 1 column that itself has n number of components within and the Vector exists in $\mathbb{R}^n$

Matrix Addition and Subtraction

Matrix addition or subtraction of two Matrices $A_{m\,n}$ and $B_{m\,n}$ is defined for Matrices that have the same size m by n.

$$S_{m\,n} = A_{m\,n} \pm B_{m\,n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{pmatrix}$$

$$S = position\ i\ j \pm position\ i\ j\ in\ position\ i\ j = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} & \dots & a_{2n} \pm b_{2n} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} & \dots & a_{3n} \pm b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & a_{m3} \pm b_{m3} & \dots & a_{mn} \pm b_{mn} \end{pmatrix}$$

The addition or subtraction of $A_{m\,n}$ and $B_{m\,n}$ requires the number of rows m and the number of columns n be equal.
The addition or subtraction of Matrices $A_{m\,n}$ and $B_{m\,n}$ results in a Matrix $S_{m\,n}$ whose size is m by n.

Matrix Matrix Multiplication

Matrix multiplication of two Matrices $A_{m\,n}$ and $B_{n\,r}$ is defined for Matrices that have equal inner size m by n and n by r.

$$M_{m\,r} = A_{m\,n}\, B_{n\,r} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1r} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2r} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nr} \end{pmatrix} = row\ i \times column\ j\ in\ position\ i\ j$$

$$M = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1} & a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1n}b_{n2} & \dots & a_{11}b_{1r} + a_{12}b_{2r} + \dots + a_{1n}b_{nr} \\ a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2n}b_{n1} & a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2n}b_{n2} & \dots & a_{21}b_{1r} + a_{22}b_{2r} + \dots + a_{2n}b_{nr} \\ a_{31}b_{11} + a_{32}b_{21} + \dots + a_{3n}b_{n1} & a_{31}b_{12} + a_{32}b_{22} + \dots + a_{3n}b_{n2} & \dots & a_{31}b_{1r} + a_{32}b_{2r} + \dots + a_{3n}b_{nr} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \dots + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + \dots + a_{mn}b_{n2} & \dots & a_{m1}b_{1r} + a_{m2}b_{2r} + \dots + a_{mn}b_{nr} \end{pmatrix}$$

The multiplication of $A_{m\,n}$ and $B_{n\,r}$ requires the number of columns n of $A_{m\,n}$ equal the number of rows n of $B_{n\,r}$.
The multiplication of Matrices $A_{m\,n}$ and $B_{n\,r}$ results in a Matrix $M_{m\,r}$ whose size is m by r. Note $A_{m\,n}\, B_{n\,r} \neq B_{n\,r}\, A_{m\,n}$.

Matrix Vector Multiplication

Matrix multiplication of Matrix $A_{m\,n}$ and Vector x_n is defined for Matrices that have equal inner size m by n and n by 1.

$$b_m = A_{m\,n}\, x_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}\,x_3 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + a_{23}\,x_3 + \dots + a_{2n}v_n \\ a_{31}v_1 + a_{32}v_2 + a_{33}\,x_3 + \dots + a_{3n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + a_{m3}\,x_3 + \dots + a_{mn}v_n \end{pmatrix}$$

The multiplication of $A_{m\,n}$ and x_n requires the number of columns n of $A_{m\,n}$ equal the number of rows n of x_n.
The multiplication of Matrix $A_{m\,n}$ and Vector x_n results in a Vector b_m whose size is m by 1.

Scalar Matrix Multiplication

Scalar Matrix multiplication of a Scalar c and Matrix $A_{m\,n}$ is defined for Matrices with any size m by n.

$$C_{m\,n} = c\,A_{m\,n} = c\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} c\,a_{11} & c\,a_{12} & c\,a_{13} & \dots & c\,a_{1n} \\ c\,a_{21} & c\,a_{22} & c\,a_{23} & \dots & c\,a_{2n} \\ c\,a_{31} & c\,a_{32} & c\,a_{33} & \dots & c\,a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c\,a_{m1} & c\,a_{m2} & c\,a_{m3} & \dots & c\,a_{mn} \end{pmatrix}$$

The multiplication of a Scalar c and a Matrix $A_{m\,n}$ results in a Matrix $C_{m\,n}$ whose size is m by n.

Scalar Vector Multiplication

Scalar Matrix multiplication of a Scalar c and Vector x_n is defined for Vectors with any size n by 1.

$$b_n = c\,x_n = c\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ \vdots \\ c\,x_n \end{pmatrix}$$

The multiplication of a Scalar c and a Vector x_n results in a Vector b_n whose size is n by 1.

Identity Matrix

The Identity Matrix I is a special type of Square Matrix that has the property of being the matrix multiplication identity

$$A_{n\,n}\,I_{n\,n} = I_{n\,n}\,A_{n\,n} = A_{n\,n}$$

For the Identity Matrix $I_{n\,n}$ to have such a property it is necessary for the form of 1 on its diagonal and 0 elsewhere.

$$I_{n\,n} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$For\ the\ 2\ by\ 2\ matrix\ I_{2\,2} \qquad I_{2\,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$For\ the\ 3\ by\ 3\ matrix\ I_{3\,3} \qquad I_{3\,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When the identity $I_{n\,n}$ is written into its Vector Linear Combination Form, it is a combination of Elementary Vectors e_j.

$$I_{n\,n} = 1\,e_1 + 1\,e_2 + 1\,e_3 + \cdots + 1\,e_n$$

$$I_{n\,n} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + 1\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

The Elementary Basis Vector e_j is a vector with a value of 1 in the j row position and a value of 0 in all other positions.

$$e_j = \begin{pmatrix} Row\ 1 \\ Row\ 2 \\ Row\ 3 \\ \vdots \\ Row\ j-1 \\ Row\ j \\ Row\ j+1 \\ \vdots \\ Row\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad and\ any\ vector \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_m \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + v_m\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Equivalent System of Equations and Elementary Row Operations

An Equivalent System of Equations is any collection of equations that will produce the exact solution set as the original system of equations. Equivalent Systems of Equations are often formed in the process of determining the solution set. The solution set is also an Equivalent System of Equations but in a form that the solution set is obvious. A Nonequivalent System of Equations will not produce the exact solution set as the original system of equations and should not be used. In order to keep an Equivalent System of Equations, it is necessary as algebra to use only Elementary Row Operations.

$c\,R_i + R_j \to R_j \quad \textit{Row Combination} \qquad c\,R_i \to R_i \quad \textit{Row Scaling} \qquad R_i \leftrightarrow R_j \quad \textit{Row Interchange}$

Elementary Row Operations

Elementary Row Operations are any algebra processes that will retain an Equivalent System of Equations. There are three algebra processes that are Elementary Row Operations: Row Combination, Row Scaling, and Row Interchange.

Row Combination

Row Combination is a scalar multiple of one row added to a second row and placed into the position of the second row.

$c\,R_i + R_j \to R_j \qquad \textit{Elementary Row Operation Notation for a scalar mutiple of row i added to row j}$

The Row Combination on matrix A has no effect on the determinant for the resulting matrix B so that $\det B = \det A$.

The Row Combination result of scalar multiple c of row R_i added to row R_j and placed into row R_j on a general matrix is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{c\,R_i + R_j \to R_j} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{j1} + c\,a_{i1} & a_{j2} + c\,a_{i2} & a_{j3} + c\,a_{i3} & \cdots & a_{jn} + c\,a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Row Scaling

Row Scaling is a scalar multiple of one row replaced back into the position of the row creating a multiple of the row.

$c\,R_i \to R_i \qquad \textit{Elementary Row Operation Notation for a scalar mutiple of row i placed into row i}$

The Row Scaling on matrix A has an effect on the determinant for the resulting matrix B so that $\det B = c\ \det A$.

The Row Scaling result of scalar multiple c of row R_i replaced back into row R_i on a general matrix is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{c\,R_i \to R_i} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c\,a_{i1} & c\,a_{i2} & c\,a_{i3} & \cdots & c\,a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Row Interchange

Row Interchange is a row put into the position of a second row and the second row placed into the position of the row.

$R_i \leftrightarrow R_j \qquad \textit{Elementary Row Operation Notation for an interchange between row i and row j}$

The Row Combination on matrix A has an effect on the determinant for the resulting matrix B so that $\det B = -\det A$.

The Row Interchange result of row R_i placed into row R_j and row R_j placed into row R_i on a general matrix is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \xrightarrow{R_i \leftrightarrow R_j} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Row Reduction of a Matrix, Echelon Form, and Reduced Echelon Form

Row Reduction of a Matrix is the use of a certain number of Elementary Row Operations in order to reduce a matrix to a more simplified form that will have more zeros in the matrix positions. Often the final goal of Row Reduction of a Matrix is either Echelon Form or Reduced Row Echelon Form in order to determine properties or solutions for the matrix.

Echelon Form

Echelon Form is necessary to determine the column vector dependence or independence, consistency or inconsistency of solutions, number of solutions which is always either zero, one, or infinity, rank, nullity, number of pivot positions, number of free variables, the column space, dimension of the column space, dimension of the null space, dimension of the solution space, number of basis vectors in the null space, number of basis vectors in the solution space, if the matrix is one to one, if the matrix is onto, if the columns span the entire $\mathbb{R}^n$ space, and if a square matrix is invertible.

To Row Reduce a matrix through Elementary Row Operations to its equivalent Echelon Form through steps

1. If possible create a value of 1 in the a_{11} position by either interchanging two rows or scale row R_1 by value $\frac{1}{a_{11}}$. If not create a value of 1 in the a_{12} position by either interchanging two rows or scale row R_1 by value $\frac{1}{a_{12}}$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_j \quad or \quad \frac{1}{a_{11}} R_1 \rightarrow R_1} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

2. Row combine scalar $-a_{21}$ row R_1 with row R_2 followed by scalar $-a_{31}$ row R_1 with row R_3 followed by scalar $-a_{41}$ row R_1 with row R_4 continuing through scalar $-a_{m1}$ row R_1 with row R_m creating zeros under the 1.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{-a_{21}\, R_1 + R_2 \rightarrow R_2 \quad up\ through \quad -a_{m1}\, R_1 + R_m \rightarrow R_m} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

3. If possible create a value of 1 in the a_{22} position by either interchanging two rows or scale row R_2 by value $\frac{1}{a_{22}}$. If not create a value of 1 in the a_{23} position by either interchanging two rows or scale row R_2 by value $\frac{1}{a_{23}}$.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_j \quad or \quad \frac{1}{a_{11}} R_1 \rightarrow R_1} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

4. Row combine scalar $-a_{32}$ row R_2 with row R_3 followed by scalar $-a_{42}$ row R_2 with row R_4 followed by scalar $-a_{52}$ row R_2 with row R_5 continuing through scalar $-a_{m2}$ row R_2 with row R_m creating zeros under the 1.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{-a_{32}\, R_2 + R_3 \rightarrow R_3 \quad up\ through \quad -a_{m2}\, R_2 + R_m \rightarrow R_m} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

5. Continue repeating the process up through column C_n to put the matrix in its equivalent Echelon Form.

$$\begin{pmatrix} 1\ or\ 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1\ or\ 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 1\ or\ 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1\ or\ 0 \end{pmatrix}$$

Results of Echelon Form

The Echelon Form of a Matrix produces a large amount of information about the properties of a matrix. Echelon Form is created when the matrix has an upper left to lower right configuration for its Pivot Positions or its leading nonzero term in each row. If the first nonzero term in each row is any number of columns to the right of the column for the first nonzero term in the row directly above it, the matrix is in Echelon Form. In Echelon Form all complete rows of zeros are at the bottom of the matrix. Any matrix can be converted into Echelon Form through Elementary Row Operations.

A Pivot Position is the first nonzero term in any given row of a matrix once it is in Echelon Form. If the value of the Pivot Position in a given row is a 1, it is also known as a Leading One. A complete row of zeros will not have a Pivot Position. A Free Variable Position is in all columns that do not contain a Pivot Position within a matrix once it is in Echelon Form.

Echelon Form has an upper left to lower right structure with the Pivot Position of each row any number of columns to the right of the Pivot Position in the row directly above it or any number of columns to the left of the Pivot Position in the row directly below it. All complete rows of zeros should be located at the bottom of a matrix in Echelon Form.

$$\text{Echelon Form for } A_{m\,n} = \begin{pmatrix} \text{Row 1 Pivot Position in column } C_a \text{ where } 0 \le a \le n \\ \text{Row 2 Pivot Position in column } C_b \text{ where } a < b \le n \\ \text{Row 3 Pivot Position in column } C_c \text{ where } b < c \le n \\ \text{Row 4 Pivot Position in column } C_d \text{ where } c < d \le n \\ \vdots \\ \text{All Complete Rows of Zeros located at the bottom} \end{pmatrix}$$

Echelon Form is a structure with terms having a value of zero below all Pivot Positions but with terms having any value above all Pivot Positions. Terms not located above or below a Pivot Position can have any value.

The Solution Space is the entire collection of x_i values that make a Linear System of Equations hold true.

$$\begin{matrix} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \cdots + a_{1n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \cdots + a_{2n}\,x_n = b_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 + \cdots + a_{3n}\,x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + \cdots + a_{mn}\,x_n = b_m \end{matrix}$$

The Solution Space has either Zero Solutions and Inconsistent, exactly One Solution, or an Infinite Number of Solutions.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \text{Zero Solutions and Inconsistent} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} \quad \text{Exactly One Solution}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} + x_{f1} \begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2} \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3} \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq} \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix} \quad \text{Infinite Number of Solutions}$$

Zero Solutions occurs when a row of zeros equals a nonzero value within the Reduced Echelon Augmented Matrix Form. Exactly One Solution occurs when the system is consistent but with no Free Variables in the Reduced Echelon Form. Infinite Solutions occur when the system is consistent but with at least one Free Variable in the Reduced Echelon Form.

Properties for the System of Equations and its Solutions except for the Solution values are found by the Echelon Form of:

$$A_{m\,n} = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

For the solutions values themselves, it is best to continue the row reduction steps all the way to Reduced Echelon Form.

A System of Equations has Matrix Properties that are determined by the Echelon Form of the Matrix.

$$\begin{matrix} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \cdots + a_{1n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \cdots + a_{2n}\,x_n = b_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 + \cdots + a_{3n}\,x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + \cdots + a_{mn}\,x_n = b_m \end{matrix} \qquad \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

Echelon Form has terms with zero value below all Pivot Positions but with terms of any value above all Pivot Positions.

$$Echelon\ Form\ for\ A_{m\,n} = \begin{pmatrix} Row\ 1\ Pivot\ Position\ in\ column\ C_a\ where\ 0 \le a \le n \\ Row\ 2\ Pivot\ Position\ in\ column\ C_b\ where\ a < b \le n \\ Row\ 3\ Pivot\ Position\ in\ column\ C_c\ where\ b < c \le n \\ Row\ 4\ Pivot\ Position\ in\ column\ C_d\ where\ c < d \le n \\ \vdots \\ All\ Complete\ Rows\ of\ Zeros\ located\ at\ the\ bottom \end{pmatrix}$$

In Echelon Form the matrix should contain a nonzero value in each Pivot Position x_p and any nonzero value in each Free Variable Position x_f. The Pivot Positions x_p and the Free Variable Positions x_f determine the properties of the matrix.

Number of Pivot Positions

A Pivot Position is the first nonzero term in any given row of a matrix once it is in Echelon Form. If the value of the Pivot Position in a given row is a 1, it is also known as a Leading One. A complete row of zeros will not have a Pivot Position. Number of Pivot Positions in an Echelon Form matrix can be counted and can be as large as the number of columns n.

Number of Free Variables

A Free Variable Position is in all columns that do not contain a Pivot Position within a matrix once it is in Echelon Form. Number of Free Variables in an Echelon Form matrix can be counted and can be as large as the number of columns n.

Consistency of Solutions

A Linear System of Equations is Consistent if at least one solution exists to the system. It is possible for a Linear System of Equations to have No Solutions or be Inconsistent, and this will occur if the Echelon Form of an Augmented Matrix has at least one complete row of zeros in the matrix but a nonzero value within the same row in the augmented vector.

$$System\ is\ Inconsistent\ and\ has\ No\ Solutions\ or\ Zero\ Solutions\ if\ \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & nonzero\ entry \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$Otherwise\ System\ is\ Consistent\ and\ will\ have\ either\ One\ Solution\ or\ an\ Infinite\ Number\ of\ Solutions$$

Number of Solutions

A Linear System of Equations can have either Zero Solutions, Exactly One Solution, or an Infinite Number of Solutions
Zero Solutions or Inconsistent occurs with a row of zeros in the matrix and a nonzero entry in the augmented vector.
One Solution if Consistent with no Free Variables and an Infinite Number of Solutions if Consistent with Free Variables

$$Zero\ Solutions\ or\ Inconsistent\ if\ a\ Complete\ Row\ of\ Zeros\ Equal\ to\ a\ Nonzero\ Entry$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & nonzero\ entry \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$One\ Solution\ if\ no\ Free\ Variables\ and\ Consistent$$
$$Infinity\ Solutions\ if\ Free\ Variables\ and\ Consistent$$
$$Infinity\ Solutions\ if\ Free\ Variables\ and\ Consistent$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & zero\ entry \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

Column Vector Independence or Dependence

Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a dependent column and each column that contains a Pivot Position x_p is an independent column and the equivalent column from the original matrix is one of the Column Basis Vectors. If the matrix contains a Pivot Position in every column, the matrix has an independent column set. If the matrix has one or more free variables, the matrix has a dependent column set where each column is a linear combination of the others and each column is dependent on the others. To make a dependent column set independent, eliminate all columns from the original matrix that resulted in a Free Variable Position in the Echelon Form of the matrix. For a Square Matrix each column with a Pivot Position is equivalent to a row with a Pivot Position.

Column Space

The Column Space is the set of columns formed by the linear combination of all Column Basis vectors or all independent column vectors in the original matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f represents a dependent column from the original matrix A and each column that contains a Pivot Position x_p represents an independent column from the original matrix A and the equivalent column from the original matrix A is one of the Column Space Basis Vectors. The remaining dependent columns can be ignored.

To form the Basis Vectors to the Column Space of a matrix A, row reduce the matrix to Echelon Form. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f represents a dependent column from the original matrix A and each column that contains a Pivot Position x_p represents an independent column from the original matrix A. Return to the original matrix A and pick out each column vector that resulted in a Pivot Position column in the Echelon Form of the matrix. Collect the resulting vectors together to form the Basis Vectors to the Column Space.

Number of Basis Vectors to the Column Space

The Number of Basis Vectors to the Column Space is the number of Independent Column Vectors in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a dependent column and each column that contains a Pivot Position x_p is an independent column. The number of Pivot Positions x_p is the Number of Basis Vectors to the Column Space. The Basis Vectors themselves will be the equivalent Pivot Position columns in the original matrix.

Dimension of the Column Space

The Dimension of the Column Space is the number of Independent Column Vectors in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a dependent column and each column that contains a Pivot Position x_p is an independent column. The number of Pivot Positions x_p is the Dimension of the Column Space. Maximum Dimension of the Column Space is columns in a matrix n and occurs with a Pivot Position x_p in every column.

Rank

The Rank is the Dimension of the Column Space and the number of Independent Column Vectors in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a dependent column and each column that contains a Pivot Position x_p is an independent column. The number of Pivot Positions x_p is the Rank of the matrix.

$$Rank\ A_{m\,n} + Nullity\ A_{m\,n} = Number\ of\ Pivot\ Positions + Number\ of\ Free\ Variables = n\ Number\ of\ Columns$$

Geometric Form for Span of Column Space Basis Vectors within $\mathbb{R}^{\boldsymbol{m}}$ space

Number of Pivot Positions x_p in the Echelon Form matrix is the Rank or Dimension of the Column Space of the matrix.
If the Rank or the Dimension of the Column Space is zero, the Geometric Form of the Column Space is a Single Point.
If the Rank or the Dimension of the Column Space is one, the Geometric Form of the Column Space is a Line.
If the Rank or the Dimension of the Column Space is two, the Geometric Form of the Column Space is a Plane.
If the Rank or the Dimension of the Column Space is three, the Geometric Form of the Column Space is$3\ dim\ of\ \mathbb{R}^{\boldsymbol{m}}$.
If the Rank or the Dimension of the Column Space is j, the Geometric Form of the Column Space is $j\ dim\ of\ \mathbb{R}^{\boldsymbol{m}}$.
If the Rank or the Dimension of the Column Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Column Space is all of $\mathbb{R}^m$. This will occur if there is a Pivot Position x_p in every column.

Number of Basis Vectors to the Null Space

The Number of Basis Vectors to the Null Space is the number of Free Variables in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a free variable column and each column that contains a Pivot Position x_p is a pivot position column. The number of Free Variables x_f is the Number of Basis Vectors to the Null Space. The Null Space is the solution for the homogenous linear system of equations equal to zero and centered on the origin.

Dimension of the Null Space

The Dimension of the Null Space is the number of Free Variable columns in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a free variable column and each column that contains a Pivot Position x_p is a pivot position column. The number of Free Variables x_f is the Dimension of the Null Space. Maximum Dimension of the Null Space is columns in a matrix n and occurs with a Free Variable x_f in every column.

Nullity

The Nullity is the Dimension of the Null Space and the number of Free Variable columns in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a free variable column and each column that contains a Pivot Position x_p is a pivot position column. The number of Free Variables x_f is the Nullity of the matrix.

$$Rank\ A_{m\,n} + Nullity\ A_{m\,n} = Number\ of\ Pivot\ Positions + Number\ of\ Free\ Variables = n\ Number\ of\ Columns$$

Geometric Form for Span of Null Space Basis Vectors

Number of Free Variables x_f in the Echelon Form matrix is the Nullity or Dimension of the Null Space of the matrix.
If the Nullity or the Dimension of the Null Space is zero, the Geometric Form of the Null Space is a Single Point.
If the Nullity or the Dimension of the Null Space is one, the Geometric Form of the Null Space is a Line.
If the Nullity or the Dimension of the Null Space is two, the Geometric Form of the Null Space is a Plane.
If the Nullity or the Dimension of the Null Space is three, the Geometric Form of the Null Space is $3\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Null Space is j, the Geometric Form of the Null Space is $j\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Null Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Null Space is all of $\mathbb{R}^n$. This will occur if there is a Free Variable x_f in every column.

Number of Basis Vectors to the Solution Space

The Number of Basis Vectors to the Solution Space is the number of Free Variables in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a free variable column and each column that contains a Pivot Position x_p is a pivot position column. The number of Free Variables x_f is the Number of Basis Vectors to the Solution Space. The Solution Space is centered on any point, with the special Null Space centered on the origin.

Dimension of the Solution Space

The Dimension of the Solution Space is the number of Free Variable columns in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a free variable column and each column that contains a Pivot Position x_p is a pivot position column. The number of Free Variables x_f is the Dimension of the Solution Space. Maximum Dimension of the Solution Space is columns in a matrix n and occurs with a Free Variable x_f in every column.

Geometric Form for Span of Solution Space Basis Vectors

Number of Free Variables x_f in the Echelon Form matrix is the Nullity or Dimension of the Solution Space of the matrix.
If the Nullity or the Dimension of the Solution Space is zero, the Geometric Form of the Solution Space is a Single Point.
If the Nullity or the Dimension of the Solution Space is one, the Geometric Form of the Solution Space is a Line.
If the Nullity or the Dimension of the Solution Space is two, the Geometric Form of the Solution Space is a Plane.
If the Nullity or the Dimension of the Solution Space is three, the Geometric Form of the Solution Space is $3\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Solution Space is j, the Geometric Form of the Solution Space is $j\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Solution Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Solution Space is all of $\mathbb{R}^n$. This will occur if there is a Free Variable x_f in every column.

One to One Matrix

A One to One Matrix has a Pivot Position x_p in every column. A One to One Matrix has at most one solution vector in the domain space $\mathbb{R}^n$ for each vector in the codomain space $\mathbb{R}^m$ to the system of $A\,\vec{x} = \vec{b}$. A One to One Matrix has either zero solutions to the inconsistent system $A\,\vec{x} = \vec{b}$ or exactly one solution to the consistent system $A\,\vec{x} = \vec{b}$.

A One to One Matrix has exactly one solution to the always consistent homogenous system $A\,\vec{x} = \vec{0}$. The one solution will be the trivial solution, or the all zero solution vector $\vec{x} = \vec{0}$.

A rectangular matrix $A_{m\,n}$ that has more columns than rows $n > m$ cannot be a One to One Matrix.

Onto Matrix

An Onto Matrix has a Pivot Position x_p in every row. An Onto Matrix has at least one solution vector in the domain space $\mathbb{R}^n$ for each vector in the codomain space $\mathbb{R}^m$ to the system of $A\,\vec{x} = \vec{b}$. An Onto Matrix has either one solution to the consistent system $A\,\vec{x} = \vec{b}$ with no free variables in the Echelon Form or an infinite number of solutions to the consistent system $A\,\vec{x} = \vec{b}$ with at least one free variable in the Echelon Form.

An Onto Matrix has an infinite number of solutions to the always consistent homogenous system $A\,\vec{x} = \vec{0}$. The infinite number of solutions will always include the trivial solution, or the all zero solution vector $\vec{x} = \vec{0}$.

A rectangular matrix $A_{m\,n}$ that has more rows than columns $m > n$ cannot be an Onto Matrix.

One to One and Onto Matrix

A One to One and Onto Matrix has a Pivot Position x_p in every column and has a Pivot Position x_p in every row. A One to One and Onto Matrix has exactly one solution vector in the domain space $\mathbb{R}^n$ for each vector in the codomain space $\mathbb{R}^m$ to the system of $A\,\vec{x} = \vec{b}$ which will always be consistent for a One to One and Onto Matrix.

A One to One and Onto Matrix has exactly one solution to the always consistent homogenous system $A\,\vec{x} = \vec{0}$. The one solution will be the trivial solution, or the all zero solution vector $\vec{x} = \vec{0}$.

A Rectangular Matrix $A_{m\,n}$ with $m \neq n$ is either One to One, Onto, or neither.
A Square Matrix $A_{n\,n}$ with $m = n$ is either both One to One and Onto, or neither.

A rectangular matrix $A_{m\,n}$ that has more columns than rows $n > m$ cannot be a One to One Matrix with a Pivot Position x_p in every column but may be an Onto Matrix if it has a Pivot Position x_p in every row, or may be neither One to One nor Onto if it does not have a Pivot Position x_p in every row.

A rectangular matrix $A_{m\,n}$ that has more rows than columns $m > n$ cannot be an Onto Matrix with a Pivot Position x_p in every row but may be a One to One Matrix if it has a Pivot Position x_p in every column, or may be neither One to One nor Onto if it does not have a Pivot Position x_p in every column.

A square matrix $A_{n\,n}$ can be both a One to One and Onto Matrix if it has a Pivot Position x_p in every column and row, or may be neither One to One nor Onto if it does not have a Pivot Position x_p in every column and row.

Invertible Matrix

An Invertible Matrix is a Square Matrix $A_{n\,n}$ that is both One to One and Onto
An Invertible Matrix is a Square Matrix $A_{n\,n}$ with a Pivot Position x_p in every column and a Pivot Position x_p in every row.
An Invertible Matrix is a Square Matrix $A_{n\,n}$ with an Inverse Matrix $A_{n\,n}^{-1}$ defined for it as $A_{n\,n}\,A_{n\,n}^{-1} = A_{n\,n}^{-1}\,A_{n\,n} = I_{n\,n}$.

Reduced Echelon Form

Reduced Echelon Form is necessary to determine the null space, the null space basis vectors, the solution space, or the solution space basis vectors of the augmented matrix and the inverse or the eigenvectors of a square matrix.

To Row Reduce a matrix through Elementary Row Operations to its equivalent Reduced Echelon Form through steps

1. If possible create a value of 1 in the a_{11} position by either interchanging two rows or scale row R_1 by value $\frac{1}{a_{11}}$.
   If not create a value of 1 in the a_{12} position by either interchanging two rows or scale row R_1 by value $\frac{1}{a_{12}}$.
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_j \;\; or \;\; \frac{1}{a_{11}} R_1 \to R_1} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$
2. Row combine scalar $-a_{21}$ row R_1 with row R_2 followed by scalar $-a_{31}$ row R_1 with row R_3 followed by scalar $-a_{41}$ row R_1 with row R_4 continuing through scalar $-a_{m1}$ row R_1 with row R_m creating zeros under the 1.
$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{-a_{21}\, R_1 + R_2 \to R_2 \;\; up\ through \;\; -a_{m1}\, R_1 + R_m \to R_m} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$
3. If possible create a value of 1 in the a_{22} position by either interchanging two rows or scale row R_2 by value $\frac{1}{a_{22}}$.
   If not create a value of 1 in the a_{23} position by either interchanging two rows or scale row R_2 by value $\frac{1}{a_{23}}$.
$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_j \;\; or \;\; \frac{1}{a_{11}} R_1 \to R_1} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$
4. Row combine scalar $-a_{32}$ row R_2 with row R_3 followed by scalar $-a_{42}$ row R_2 with row R_4 followed by scalar $-a_{52}$ row R_2 with row R_5 continuing through scalar $-a_{m2}$ row R_2 with row R_m creating zeros under the 1.
$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{-a_{32}\, R_2 + R_3 \to R_3 \;\; up\ through \;\; -a_{m2}\, R_2 + R_m \to R_m} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{m3} & \dots & a_{mn} \end{pmatrix}$$
5. Continue repeating the process up through column C_n to put the matrix in its equivalent Echelon Form.
6. Row combine scalar $-a_{m-1\,n}$ row R_m with row R_{m-1} followed by scalar $-a_{m-2\,n}$ row R_m with row R_{m-2} continuing through scalar $-a_{1\,n}$ row R_m with row R_1 creating zeros above the 1.
$$\begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{m3} & \dots & a_{mn} \end{pmatrix} \xrightarrow{-a_{m-1\,n}\, R_m + R_{m-1} \to R_{m-1} \;\; to \;\; -a_{m-2\,n}\, R_m + R_{m-2} \to R_{m-2}} \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & 0 \\ 0 & 1 & a_{23} & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$
7. Continue repeating the process up through column C_1 to put the matrix in its equivalent Reduced Echelon Form.
$$\begin{pmatrix} 1\ or\ 0 & 0 & 0 & \dots & 0 \\ 0 & 1\ or\ 0 & 0 & \dots & 0 \\ 0 & 0 & 1\ or\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1\ or\ 0 \end{pmatrix}$$

Results of Reduced Echelon Form

The Reduced Echelon Form of a Matrix produces a large amount of information about the solutions of a matrix. Reduced Echelon Form is created when the matrix has an upper left to lower right configuration for its Pivot Positions or its leading nonzero term in each row but each Pivot Position has a value of exactly one while all other terms in the matrix are exactly zero. If the first nonzero term in each row is a value of one and any number of columns to the right of the column for the first nonzero term which should also be a value of one in the row directly above it, and all other terms in the same column as each first nonzero term in each row are themselves all zeros, the matrix is in Reduced Echelon Form. In Reduced Echelon Form all complete rows of zeros are at the bottom of the matrix. Any matrix can be converted into Echelon Form through Elementary Row Operations. Terms not above or below a Pivot Position can be any value.

A Pivot Position is the first nonzero term in any given row of a matrix. If the value of the Pivot Position in a given row is a 1, the Pivot Position is also known as a Leading One. By definition a complete row of zeros has no Pivot Position.
A Free Variable Position is in all columns that do not contain a Pivot Position within a matrix once it is in Echelon Form.

Reduced Echelon Form has an upper left to lower right structure with the Pivot Position of each row having a value of 1 and located any number of columns to the right of the Pivot Position having a value of one in the row directly above it or any number of columns to the left of the Pivot Position having a value of one in the row directly below it. All complete rows of zeros should be located at the bottom of a matrix in Reduced Echelon Form.

$$\text{Reduced Echelon Form for } A_{m\,n} = \begin{pmatrix} \text{Row 1 Pivot Position value 1 with zeros in column } C_a \text{ where } 0 \le a \le n \\ \text{Row 2 Pivot Position value 1 with zeros in column } C_b \text{ where } a < b \le n \\ \text{Row 3 Pivot Position value 1 with zeros in column } C_c \text{ where } b < c \le n \\ \text{Row 4 Pivot Position value 1 with zeros in column } C_d \text{ where } c < d \le n \\ \vdots \\ \text{All Complete Rows of Zeros located at the bottom} \end{pmatrix}$$

Reduced Echelon Form is a structure of all Pivot Positions having a value of exactly one with terms having a value of zero below all Pivot Positions as well as with terms having a value of zero above all Pivot Positions. Terms not located above or below a Pivot Position can have any value and are the Free Variable Positions within the Reduced Echelon Form.

The Identity Matrix is a Square Matrix that is in Reduced Echelon Form without any rows of zeros.
The Reduced Echelon Form of a matrix has all Pivot Position columns as Elementary Basis Vectors e_j.

The Solution Space is the entire collection of x_i values that make a Linear System of Equations hold true.

$$\begin{matrix} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \cdots + a_{1n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \cdots + a_{2n}\,x_n = b_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 + \cdots + a_{3n}\,x_n = b_3 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + \cdots + a_{mn}\,x_n = b_m \end{matrix}$$

Consistency of Solutions

A Linear System of Equations is Consistent if at least one solution exists to the system. It is possible for a Linear System of Equations to have No Solutions or be Inconsistent, and this will occur if the Echelon Form of an Augmented Matrix has at least one complete row of zeros in the matrix but a nonzero value within the same row in the augmented vector.

$$\text{System is Inconsistent and has No Solutions or Zero Solutions if} \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \text{nonzero entry} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array}\right)$$

$$\text{Otherwise System is Consistent and will have either One Solution or an Infinite Number of Solutions}$$

Number of Solutions

Zero Solutions or Inconsistent occurs with a row of zeros in the matrix and a nonzero entry in the augmented vector. Exactly One Solution occurs when the system is consistent but with no Free Variables in the Reduced Echelon Form. Infinite Solutions occur when the system is consistent but with at least one Free Variable in the Reduced Echelon Form.

$$\begin{gathered}\textit{Zero Solutions or Inconsistent if a}\\ \textit{Complete Row of Zeros Equal to a Nonzero Entry}\end{gathered}$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & nonzero\ entry \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$\begin{gathered}\textit{One Solution if no Free Variables and Consistent}\\ \textit{Infinity Solutions if Free Variables and Consistent}\end{gathered}$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & zero\ entry \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

Solution Set

A System of Equations has a Solution Set that is determined by the Reduced Echelon Form of the Augmented Matrix.

$$\begin{array}{c} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \cdots + a_{1n}\,x_n = b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \cdots + a_{2n}\,x_n = b_2 \\ a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 + \cdots + a_{3n}\,x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + \cdots + a_{mn}\,x_n = b_m \end{array}$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

In Reduced Echelon Form the matrix should contain a value of 1 in each Pivot Position x_p and any nonzero value in each Free Variable Position x_f. The Augmented Matrix Form can be rewritten back into Linear System of Equations Form:

$$\begin{array}{c} 1\,x_{p1} + c_{11}\,x_{f1} + c_{12}\,x_{f2} + \cdots + c_{1q}\,x_{fq} = g_1 \\ 1\,x_{p2} + c_{21}\,x_{f1} + c_{22}\,x_{f2} + \cdots + c_{2q}\,x_{fq} = g_2 \\ 1\,x_{p3} + c_{31}\,x_{f1} + c_{32}\,x_{f2} + \cdots + c_{3q}\,x_{fq} = g_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ 1\,x_{pr} + c_{r1}\,x_{f1} + c_{r2}\,x_{f2} + \cdots + c_{rq}\,x_{fq} = g_r \end{array}$$

The resulting Linear System of Equations can then be solved for the Pivot Positions x_p in terms of the Free Variables x_f.

$$\begin{array}{c} x_{p1} = g_1 - c_{11}\,x_{f1} - c_{12}\,x_{f2} - \cdots - c_{1q}\,x_{fq} \\ x_{p2} = g_2 - c_{21}\,x_{f1} - c_{22}\,x_{f2} - \cdots - c_{2q}\,x_{fq} \\ x_{p3} = g_3 - c_{31}\,x_{f1} - c_{32}\,x_{f2} - \cdots - c_{3q}\,x_{fq} \\ \vdots \quad \vdots \quad \vdots \qquad \vdots \qquad \vdots \\ x_{pr} = g_r - c_{r1}\,x_{f1} - c_{r2}\,x_{f2} - \cdots - c_{rq}\,x_{fq} \end{array}$$

Collecting the Pivot Positions x_p in terms of the Free Variables x_f together with the Free Variable x_f relations produces

$$\begin{array}{c} x_{p1} = g_1 - c_{11}\,x_{f1} - c_{12}\,x_{f2} - \cdots - c_{1q}\,x_{fq} \\ x_{p2} = g_2 - c_{21}\,x_{f1} - c_{22}\,x_{f2} - \cdots - c_{2q}\,x_{fq} \\ x_{p3} = g_3 - c_{31}\,x_{f1} - c_{32}\,x_{f2} - \cdots - c_{3q}\,x_{fq} \\ \vdots \quad \vdots \quad \vdots \qquad \vdots \qquad \vdots \\ x_{pr} = g_r - c_{r1}\,x_{f1} - c_{r2}\,x_{f2} - \cdots - c_{rq}\,x_{fq} \end{array} \qquad \begin{array}{c} x_{f1} = x_{f1} \\ x_{f2} = x_{f2} \\ x_{f3} = x_{f3} \\ \vdots \quad \vdots \\ x_{fq} = x_{fq} \end{array}$$

Since every variable x_i is either a Pivot Position or a Free Variable, the Solution Linear System of Equations can be rewritten in the Vector Linear Combination Form with the Free Variables x_f as scalar multiples for the Solution Space

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} + x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}$$

Solution Space

The Solution Space is the set of all x_i values that solve the linear system of equations in an augmented matrix

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

The Solution Space has all h_i equal to any possible value and the Vector Linear Combination Form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} + x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}$$

Form of Solution Space

The form of the Solution Space solution set depends on the consistency of solutions and the number of solutions.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = Zero\ Solutions\ or\ Inconsistent \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} \quad Exactly\ One\ Solution$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} + x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix} \quad Infinite\ Number\ of\ Solutions$$

Solution Space Basis Vectors

The Basis Vectors of the Solution Space is the collection of vectors d_j multiplying with the Free Variables x_f.

$$Basis\ Vectors = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\}\ for\ One\ Solution\quad or\quad \left\{\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix}, \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix}, \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix}, \ldots, \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}\right\}\ for\ Infinity\ Solutions$$

Solution Space Dimension

The Solution Space Dimension is the number of Basis Vectors q in the Linear Combination Form of a Solution.
The Solution Space Dimension is the number of Free Variables x_f in the Reduced Echelon Form of the matrix.

Geometric Form for Span of Basis Vectors in the Solution Space

Number of Solution Space Basis Vectors d_j is the Nullity or Dimension of the Solution Space of the matrix.
If the Nullity or the Dimension of the Solution Space is zero, the Geometric Form of the Solution Space is a Single Point.
If the Nullity or the Dimension of the Solution Space is one, the Geometric Form of the Solution Space is a Line.
If the Nullity or the Dimension of the Solution Space is two, the Geometric Form of the Solution Space is a Plane.
If the Nullity or the Dimension of the Solution Space is three, the Geometric Form of the Solution Space is $3\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Solution Space is j, the Geometric Form of the Solution Space is $j\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Solution Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Solution Space is all of $\mathbb{R}^n$. This will occur if there is a Free Variable x_f in every column.

Null Space

The Null Space is the set of all x_i values that solve the homogenous linear system of equations in an augmented matrix

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & 0 \end{array}\right)$$

The Null Space or Homogenous Solution Space has all h_i equal to zero and the Vector Linear Combination Form is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}$$

Form of Null Space

The form of the Null Space solution set is always consistent but depends on the number of solutions.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad Exactly\ One\ Solution\ which\ is\ always\ the\ Trivial\ Solution$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix} \quad Infinite\ Number\ of\ Solutions$$

Null Space Basis Vectors

The Basis Vectors of the Null Space is the collection of vectors d_j multiplying with the Free Variables x_f.

$$Basis\ Vectors = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} \ for\ One\ Solution \quad or \quad \left\{\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix}, \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix}, \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix}, \ldots, \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}\right\} \ for\ Infinity\ Solutions$$

Null Space Dimension or Nullity

The Null Space Dimension or Nullity is the number of Basis Vectors q in the Linear Combination Form of a Null Space.
The Null Space Dimension or Nullity is the number of Free Variables x_f in the Reduced Echelon Form of the matrix.

$$Rank\ A_{m\,n} + Nullity\ A_{m\,n} = Number\ of\ Pivot\ Positions + Number\ of\ Free\ Variables = n\ Number\ of\ Columns$$

Geometric Form for Span of Null Space Basis Vectors

Number of Null Space Basis Vectors d_j is the Nullity or Dimension of the Solution Space of the matrix.
If the Nullity or the Dimension of the Null Space is zero, the Geometric Form of the Null Space is a Single Point.
If the Nullity or the Dimension of the Null Space is one, the Geometric Form of the Null Space is a Line.
If the Nullity or the Dimension of the Null Space is two, the Geometric Form of the Null Space is a Plane.
If the Nullity or the Dimension of the Null Space is three, the Geometric Form of the Null Space is $3\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Null Space is j, the Geometric Form of the Null Space is $j\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Null Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Null Space is all of $\mathbb{R}^n$. This will occur if there is a Free Variable x_f in every column.

Linear Transformations

A Linear Transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function that converts each vector u_n from the domain $\mathbb{R}^n$ to a new vector v_m from the codomain $\mathbb{R}^m$ through a series of linear relationships. If all relationships are linear, the Linear Transformation can be accomplished through the multiplication of each vector v_n from the domain $\mathbb{R}^n$ with an m by n Linear Transformation Matrix $A_{m\,n}$ producing the vector v_m from the codomain $\mathbb{R}^m$ through the series of linear relationships.

$$T: \mathbb{R}^n \to \mathbb{R}^m \qquad T(u_n) = v_m \quad \textit{is the Transformation Function notation for} \quad A_{m\,n}\, u_n = v_m$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

The components within the Linear Transformation Matrix $A_{m\,n}$ will depend on the series of linear relationships accomplished through the Linear Transformation $T: \mathbb{R}^n \to \mathbb{R}^m$. The columns within the Linear Transformation Matrix $A_{m\,n}$ will be the transformation of each of the corresponding Elementary Basis Vectors from the domain $\mathbb{R}^n$ space.

Column One c_1 of Transormation Matrix $A_{m\,n}$

$$\textit{Column One of } A_{m\,n} \qquad c_1 = \begin{pmatrix} a_{11} \\ a_{2\,1} \\ a_{3\,1} \\ \vdots \\ a_{m\,1} \end{pmatrix} = A_{m\,n}\, e_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = T(e_1)$$

Column Two c_2 of Transormation Matrix $A_{m\,n}$

$$\textit{Column Two of } A_{m\,n} \qquad c_2 = \begin{pmatrix} a_{12} \\ a_{2\,2} \\ a_{3\,2} \\ \vdots \\ a_{m\,2} \end{pmatrix} = A_{m\,n}\, e_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = T(e_2)$$

Column Three c_3 of Transormation Matrix $A_{m\,n}$

$$\textit{Column Three of } A_{m\,n} \qquad c_3 = \begin{pmatrix} a_{13} \\ a_{2\,3} \\ a_{3\,3} \\ \vdots \\ a_{m\,3} \end{pmatrix} = A_{m\,n}\, e_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = T(e_3)$$

$$\vdots$$

Column n c_n of Transormation Matrix $A_{m\,n}$

$$\textit{Column n of } A_{m\,n} \qquad c_n = \begin{pmatrix} a_{1\,n} \\ a_{2\,n} \\ a_{3\,n} \\ \vdots \\ a_{m\,n} \end{pmatrix} = A_{m\,n}\, e_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = T(e_n)$$

Vector Transformation Results Given

To determine each of the Linear Transformation Matrix $A_{m\,n}$ columns find the Linear Transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ of each of the corresponding Elementary Basis Vectors from the domain $\mathbb{R}^n$ space and use the following linear relationship

$$\textit{For m by n matrix} \qquad T(u) = T\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}\right) = u_1\, T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) + u_2\, T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) + u_3\, T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}\right) + \cdots + u_n\, T\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right)$$

$$T(u) = T\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}\right) = u_1\, T(e_1) + u_2\, T(e_2) + u_3\, T(e_3) + \cdots + u_n\, T(e_n)$$

2 by 2 Square Matrix Linear Transformations

A 2 by 2 Square Matrix Linear Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a function that converts each vector u_2 from the domain $\mathbb{R}^2$ to a new vector v_2 from the codomain $\mathbb{R}^2$ through a series of linear relationships.

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \qquad T(u_2) = v_2 \quad \textit{is the Transformation Function notation for} \quad A_{2\,2}\, u_2 = v_2$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The components and the columns within the Linear Transformation Matrix $A_{2\,2}$ will be the transformation of each of the corresponding Elementary Basis Vectors from the domain $\mathbb{R}^2$ space through the Linear Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$\textit{Column One } c_1 \textit{ of Transormation Matrix } A_{2\,2}$$

$$\textit{Column One of } A_{2\,2} \qquad c_1 = \begin{pmatrix} a_{1\,1} \\ a_{2\,1} \end{pmatrix} = A_{2\,2}\, e_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = T(e_1)$$

$$\textit{Column Two } c_2 \textit{ of Transormation Matrix } A_{2\,2}$$

$$\textit{Column Two of } A_{2\,2} \qquad c_2 = \begin{pmatrix} a_{1\,2} \\ a_{2\,2} \end{pmatrix} = A_{2\,2}\, e_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = T(e_2)$$

The Linear Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ on each vector u and w has the following linear relationships

$$T(u) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = u_1\, T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + u_2\, T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = u_1\, T(e_1) + u_2\, T(e_2)$$

$$T(w) = T\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = w_1\, T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + w_2\, T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = w_1\, T(e_1) + w_2\, T(e_2)$$

Vector Transformation Results Given

For a Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ the 2 by 2 square matrix $A_{2\,2}$ can be found by simultaneously solving the two equations for the two unknowns $T(e_1)$ and $T(e_2)$ in terms of vectors that result from the two relations

$$\textit{Given} \qquad T(u) = T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \textit{and} \qquad T(w) = T\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\textit{Simultaneously solve} \quad u_1\, T(e_1) + u_2\, T(e_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \textit{and} \quad w_1\, T(e_1) + w_2\, T(e_2) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \textit{for } T(e_1) \textit{ and } T(e_2)$$

$$\textit{Then write the matrix } A_{2\,2} \textit{ from Column One } c_1 \textit{ for } A_{2\,2} \quad c_1 = T(e_1) \quad \textit{and} \quad \textit{Column Two } c_2 \textit{ of } A_{2\,2} \quad c_2 = T(e_2)$$

Graph Transformation Results Given

For a Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ the 2 by 2 square matrix $A_{2\,2}$ can be found by column vector c_1 being the result for the transformation of the elementary vector e_1 and column vector c_2 being the result for the transformation of the elementary vector $e_2$following the consecutive series of transformation steps given graphically on a x_2 versus x_1 plane.

Given a series of statements for the transformations that will produce the resulting vector v_2 from vector u_2
Follow the series of statements for the transformation of e_1 for $T(e_1)$ and the transformation of e_2 for $T(e_2)$
Then write the matrix $A_{2\,2}$ from Column One c_1 for $A_{2\,2}$ $c_1 = T(e_1)$ and Column Two c_2 of $A_{2\,2}$ $c_2 = T(e_2)$

Standard 2 by 2 Square Matrix Transformations

The Standard 2 by 2 Square Matrix Transformations are a list of common linear transformations on a x_2 versus x_1 plane.

$$\textit{vector scale} \quad A_{2\,2} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a\,u_1 \\ a\,u_2 \end{pmatrix} = a \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\textit{vector shear or unequal scale} \quad A_{2\,2} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a\,u_1 \\ b\,u_2 \end{pmatrix}$$

$$\textit{vector projection onto } x_1 \textit{ axis} \quad A_{2\,2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \end{pmatrix}$$

$$\textit{vector projection onto } x_2 \textit{ axis} \quad A_{2\,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}$$

$$\textit{vector rotation through a counterclockwise angle } \theta \quad A_{2\,2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\textit{vector rotation through a clockwise angle } \theta \quad A_{2\,2} = \begin{pmatrix} \cos -\theta & -\sin -\theta \\ \sin -\theta & \cos -\theta \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos -\theta & -\sin -\theta \\ \sin -\theta & \cos -\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\textit{vector coordinate swap or mirror image about the } x_2 = x_1 \textit{ line} \quad A_{2\,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$$

Linear Transformations

A Linear Transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and may have the properties of One to One, Onto, or both One to One and Onto depending only on the m by n dimensions of its corresponding Transformation Matrix $A_{m\,n}$.

One to One Linear Transformation and Matrix

A One to One Matrix has a Pivot Position x_p in every column. A One to One Matrix has at most one solution vector in the domain space $\mathbb{R}^n$ for each vector in the codomain space $\mathbb{R}^m$ to the system of $A\,\vec{x} = \vec{b}$. A One to One Matrix has either zero solutions to the inconsistent system $A\,\vec{x} = \vec{b}$ or exactly one solution to the consistent system $A\,\vec{x} = \vec{b}$.

A One to One Matrix has exactly one solution to the always consistent homogenous system $A\,\vec{x} = \vec{0}$. The one solution will be the trivial solution, or the all zero solution vector $\vec{x} = \vec{0}$.

A rectangular matrix $A_{m\,n}$ that has more columns than rows $n > m$ cannot be a One to One Matrix.

Onto Linear Transformation and Matrix

An Onto Matrix has a Pivot Position x_p in every row. An Onto Matrix has at least one solution vector in the domain space $\mathbb{R}^n$ for each vector in the codomain space $\mathbb{R}^m$ to the system of $A\,\vec{x} = \vec{b}$. An Onto Matrix has either one solution to the consistent system $A\,\vec{x} = \vec{b}$ with no free variables in the Echelon Form or an infinite number of solutions to the consistent system $A\,\vec{x} = \vec{b}$ with at least one free variable in the Echelon Form.

An Onto Matrix has an infinite number of solutions to the always consistent homogenous system $A\,\vec{x} = \vec{0}$. The infinite number of solutions will always include the trivial solution, or the all zero solution vector $\vec{x} = \vec{0}$.

A rectangular matrix $A_{m\,n}$ that has more rows than columns $m > n$ cannot be an Onto Matrix.

One to One and Onto Linear Transformation and Matrix

A One to One and Onto Matrix has a Pivot Position x_p in every column and has a Pivot Position x_p in every row. A One to One and Onto Matrix has exactly one solution vector in the domain space $\mathbb{R}^n$ for each vector in the codomain space $\mathbb{R}^m$ to the system of $A\,\vec{x} = \vec{b}$ which will always be consistent for a One to One and Onto Matrix.

A One to One and Onto Matrix has exactly one solution to the always consistent homogenous system $A\,\vec{x} = \vec{0}$. The one solution will be the trivial solution, or the all zero solution vector $\vec{x} = \vec{0}$.

A Rectangular Matrix $A_{m\,n}$ with $m \neq n$ is either only One to One, only Onto, or neither, but cannot be both.
A Square Matrix $A_{n\,n}$ with $m = n$ is either both One to One and Onto, or neither One to One nor Onto.

A rectangular matrix $A_{m\,n}$ that has more columns than rows $n > m$ cannot be a One to One Matrix with a Pivot Position x_p in every column but may be an Onto Matrix if it has a Pivot Position x_p in every row, or may be neither One to One nor Onto if it does not have a Pivot Position x_p in every row.

A rectangular matrix $A_{m\,n}$ that has more rows than columns $m > n$ cannot be an Onto Matrix with a Pivot Position x_p in every row but may be a One to One Matrix if it has a Pivot Position x_p in every column, or may be neither One to One nor Onto if it does not have a Pivot Position x_p in every column.

A square matrix $A_{n\,n}$ can be both a One to One and Onto Matrix if it has a Pivot Position x_p in every column and row, or may be neither One to One nor Onto if it does not have a Pivot Position x_p in every column and row.

Matrix Operations of Transpose, Inverse, and Determinant

Matrix Operations of Transpose, Inverse, and Determinant are performed on all components within the matrix.

Matrix Transpose

The Matrix Transpose $A_{m\,n}^T$ is an operation that interchanges the rows and columns within a matrix and is defined by

$$\mathit{Tranpose\ Definition}\quad A_{m\,n}^T = B_{n\,m}\quad \mathit{such\ that\ each\ row\ i\ becomes\ column\ i\ and\ each\ column\ j\ becomes\ row\ j}$$

The Matrix Transpose has the property that performing Row Operations on the resulting Transpose of a Rectangular Matrix $A_{m\,n}^T$ is equivalent to performing Column Operations on the original Matrix $A_{m\,n}$ and can be used for working out Column Operations and Column Reduction on a matrix through the same steps as Row Operations and Row Reduction.

Matrix Transpose on a Rectangular Matrix

The Matrix Transpose on an m by n Rectangular Matrix $A_{m\,n}$ produces an n by m Rectangular Matrix $A_{n\,m}$.

$$\mathit{Transpose\ of\ Rectangular\ Matrix}\ A_{m\,n}\quad A_{m\,n}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{mn} \end{pmatrix}$$

The resulting Transpose on a Rectangular Matrix $A_{m\,n}^T$ will have the same number of columns as the number of rows in the original Matrix $A_{m\,n}$ and will have the same number of rows as the number of columns in the original Matrix $A_{m\,n}$.

The resulting Transpose on an m by n Rectangular Matrix $A_{m\,n}^T$ can be multiplied together with the original m by n Rectangular Matrix $A_{m\,n}$ in one order known as the Inner Product to produce a Symmetric Matrix $S_{n\,n}$ and in another order known as the Outer Product to produce a Symmetric Matrix $S_{m\,m}$with property for each of its entries $a_{i\,j} = a_{j\,i}$.

$$S_{n\,n} = A_{m\,n}^T A_{m\,n}\quad \mathit{will\ be\ the\ Inner\ Symmetric\ Matrix}\ S_{n\,n}\ \mathit{with\ all\ entries\ following\ the\ relation}\ a_{i\,j} = a_{j\,i}$$
$$S_{m\,m} = A_{m\,n} A_{m\,n}^T\quad \mathit{will\ be\ the\ Outer\ Symmetric\ Matrix}\ S_{m\,m}\ \mathit{with\ all\ entries\ following\ the\ relation}\ a_{i\,j} = a_{j\,i}$$

Matrix Transpose on a Square Matrix

The Matrix Transpose on an n by n Square Matrix $A_{n\,n}$ produces an n by n Square Matrix $A_{n\,n}$.

$$\mathit{Transpose\ of\ Square\ Matrix}\ A_{n\,n}\quad A_{n\,n}^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \dots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{pmatrix}$$

The resulting Transpose on a Square Matrix $A_{n\,n}^T$ will have the same main diagonal entries $a_{11}\ a_{22}\ a_{33} \dots a_{nn}$ as the original Matrix $A_{n\,n}$ but all other entries will interchange as a mirror image configuration around the main diagonal.

The resulting Transpose on an n by n Square Matrix $A_{n\,n}^T$ can be multiplied together with the original n by n Square Matrix $A_{n\,n}$ in either order to produce a Symmetric Matrix $S_{n\,n}$ with the property for each of its entries $a_{i\,j} = a_{j\,i}$.

$$S_{n\,n} = A_{n\,n}^T A_{n\,n} = A_{n\,n} A_{n\,n}^T\ \mathit{will\ be\ the\ Symmetric\ Matrix}\ S_{n\,n}\ \mathit{with\ all\ entries\ following\ the\ relation}\ a_{i\,j} = a_{j\,i}$$

Matrix Algebra and Operation Effects on the Transpose

Matrix and Algebra Operations on matrices A and B have the following effect on the Transpose of resulting matrix D.

The Scalar Matrix Multiplication $D = c\,A$ results in	$D^T = (c\,A)^T = c\,A^T$	*or*	$A^T = \frac{1}{c^n} D^T$
The Matrix Matrix Multiplication $D = A\,B$ results in	$D^T = (A\,B)^T = B^T A^T$	*or*	$A^T = (B^T)^{-1} D^T$
The Matrix Matrix Addition $D = A + B$ results in	$D^T = (A+B)^T = A^T + B^T$	*or*	$A^T = D^T - B^T$
The Matrix Matrix Subtraction $D = A - B$ results in	$D^T = (A-B)^T = A^T - B^T$	*or*	$A^T = D^T + B^T$
The Matrix Transpose $D = A^T$ results in	$D^T = (A^T)^T = A$	*or*	$(A^T)^T = A = D$
The Matrix Inverse $D = A^{-1}$ results in	$D^T = (A^{-1})^T = (A^T)^{-1}$	*or*	$A^T = (D^T)^{-1}$

Total Operation $\left(c\,A_1\,A_2 \dots A_p\,B_1^T\,B_2^T \dots B_q^T\,D_1^{-1}\,D_2^{-1} \dots D_r^{-1}\right)^T = c\,A_1^T\,A_2^T \dots A_p^T B_1\,B_2 \dots B_q (D_1^T)^{-1}\,(D_2^T)^{-1} \dots (D_r^T)^{-1}$

Matrix Inverse

The Matrix Inverse $A_{n\,n}^{-1}$ is a Square Matrix that is the multiplication inverse of an original Square Matrix $A_{n\,n}$ so that the two matrices multiplied with each other in either order will produce the multiplication Identity Matrix $I_{n\,n}$ as defined by

$$A_{n\,n}^{-1}\,A_{n\,n} = A_{n\,n}\,A_{n\,n}^{-1} = I_{n\,n}$$

$$A_{n\,n}^{-1}\,A_{n\,n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = I_{n\,n}$$

$$A_{n\,n}\,A_{n\,n}^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = I_{n\,n}$$

Matrix Algebra and Operation Effects on the Inverse

Matrix and Algebra Operations on matrices A and B have the following effect on the Inverse of resulting matrix D.

The Scalar Matrix Multiplication $D = c\,A$ results in	$D^{-1} = (c\,A)^{-1} = \frac{1}{c}\,A^{-1}$	*or*	$A^{-1} = c\,D^{-1}$
The Matrix Matrix Multiplication $D = A\,B$ results in	$D^{-1} = (A\,B)^{-1} = B^{-1}\,A^{-1}$	*or*	$A^{T} = B\,D^{-1}$
The Matrix Transpose $D = A^{T}$ results in	$D^{-1} = (A^{T})^{-1} = (A^{-1})^{T}$	*or*	$A^{T} = D$
The Matrix Inverse $D = A^{-1}$ results in	$D^{-1} = (A^{-1})^{-1} = A$	*or*	$A = D^{-1}$

Total Operation $\left(c\,A_1\,A_2 \dots A_p\,B_1^T\,B_2^T \dots B_q^T\,D_1^{-1}\,D_2^{-1} \dots D_r^{-1}\right)^{-1} = \frac{1}{c}\,A_1^{-1}\,A_2^{-1} \dots A_p^{-1}(B_1^{-1})^T\,(B_2^{-1})^T \dots \left(B_q^{-1}\right)^T D_1\,D_2 \dots D_r$

Finding the Inverse of a 2×2 Square Matrix

To find the Inverse of a 2×2 Square Matrix it is possible to calculate the matrix through the following formula:

$$\mathit{Inverse\ Matrix} \quad A^{-1} = \frac{1}{a\,d - b\,c}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ \mathit{or\ Singular\ if}\ a\,d - b\,c = 0 \qquad \mathit{for\ Original\ Matrix} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Finding the Inverse of a General Square Matrix

To find the Inverse of a General Square Matrix it is necessary to Row Reduce the matrix to its Reduced Echelon Form:

1. Form the Combo Rectangular Matrix $A_{n\,2n}$ by augmenting the original Matrix $A_{n\,n}$ with the Identity Matrix $I_{n\,n}$.

$$\left(\begin{array}{ccccc|ccccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & 1 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & 0 & 1 & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & 0 & 0 & 0 & \dots & 1 \end{array}\right)$$

2. Run through the necessary steps of Row Reduction on the entire matrix to produce a Reduced Row Echelon Form for just the Square Matrix $A_{n\,n}$ on the left side of the Combo Rectangular Matrix $A_{n\,2n}$.

$$\left(\begin{array}{ccccc|ccccc} 1\ or\ 0 & 0 & 0 & \dots & 0 & b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & 1\ or\ 0 & 0 & \dots & 0 & b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & 0 & 1\ or\ 0 & \dots & 0 & b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1\ or\ 0 & b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{array}\right)$$

3. If any Row Reduction steps on the left side Square Matrix $A_{n\,n}$ result in a complete row of zeros, a complete column of zeros, or a zero in any one or more of the diagonal positions $a_{11}\,a_{22}\,a_{33} \dots a_{nn}$ then the matrix is singular or noninvertible and no Inverse Matrix exists. If the Row Reduction steps on the left side Square Matrix $A_{n\,n}$ result in exactly the Identity Matrix $I_{n\,n}$ the matrix is invertible and has an Inverse Matrix $A_{n\,n}^{-1}$ defined by

$$\mathit{If\ result} \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & 1 & 0 & \dots & 0 & b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & 0 & 1 & \dots & 0 & b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{array}\right) \ \mathit{then}\ A_{n\,n}^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix}$$

Matrix Determinant

The Matrix Determinant $\det A$ is a complete permutation multiplication of all components within the rows and columns of the Matrix A. The Matrix Determinant can therefore only be defined for an n by n Square Matrix $A_{n\,n}$ as follows

For a general n by n matrix $A_{n\,n}$ the Cofactor Expansion

$$\det A_{n\,n} = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \sum_{i\ fixed,\ j=1}^{n} (-1)^{i+j}\, a_{ij}\, \det(A_{i\,j}) = \sum_{j\ fixed,\ i=1}^{n} (-1)^{i+j}\, a_{ij}\, \det(A_{i\,j})$$

Where (A_{ij}) is the square matrix formed by the removal of the ith row and the jth column of the matrix $A_{n\,n}$

For a 2 by 2 matrix $A_{2\,2}$

$$\det A = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

For a 3 by 3 matrix $A_{3\,3}$

$$\det A = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}\,(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31})$$

For a Diagonal Matrix or a Triangular Matrix $\det A = a_{11}\, a_{22}\, a_{33} \ldots a_{nn}$ *multiplication of diagonal entries*

If the matrix contains either a complete row of zeros or a complete column of zeros, its determinant will always be zero.

$\det A = 0$ *if and only if the matrix $A_{n\,n}$ reduces to a complete row of zeros or a complete column of zeros*

To find the determinant of a matrix larger than a 3 by 3, it is necessary to use the Cofactor Expansion method. In order to reduce the number of smaller matrix determinants needed in the expansion, it is recommended to expand on a row or column that has a large number of zeros within it, preferably a row or column that has only one nonzero entry resulting in only one smaller matrix determinant. If the matrix does not have a row or column with a large number of zeros, it is possible to row reduce the matrix to produce a row or a column with a large number of zeros. The Row Combination has no effect on the determinant of the resulting matrix and should be used as often as possible. The Row Scaling and Row Interchange both do have an effect on the determinant of the resulting matrix and should be used only if needed to simplify the determinant and their effect must be compensated. If at any time during row operations the matrix contains either a complete row of zeros or a complete column of zeros, its determinant will always be zero.

Row Operation Effects on the Determinant

Row Operations on a matrix A have the following effect for the Determinant of resulting matrix D.

The Row Combination $c\,R_i + R_j \to R_j$ results in	$\det D = \det A$	*or*	$\det A = \det D$
The Row Scaling $c\,R_i \to R_i$ results in	$\det D = c\,\det A$	*or*	$\det A = \frac{1}{c}\det D$
The Row Interchange $R_i \leftrightarrow R_j$ results in	$\det D = -\det A$	*or*	$\det A = -\det D$

Matrix Algebra and Operation Effects on the Determinant

Matrix and Algebra Operations on matrices A and B have the following effect on the Determinant of resulting matrix D.

The Scalar Matrix Multiplication $D = c\,A$ results in	$\det D = \det(c\,A) = c^n\,\det A$	*or*	$\det A = \frac{1}{c^n}\det D$
The Matrix Matrix Multiplication $D = A\,B$ results in	$\det D = \det A\,B = \det A\,\det B$	*or*	$\det A = \frac{\det D}{\det B}$
The Matrix Transpose $D = A^T$ results in	$\det D = \det A^T = \det A$	*or*	$\det A = \det D$
The Matrix Inverse $D = A^{-1}$ results in	$\det D = \det A^{-1} = \frac{1}{\det A}$	*or*	$\det A = \frac{1}{\det D}$

Total Operation $\det(c\,A_1\,A_2 \ldots A_p\,B_1^T\,B_2^T \ldots B_q^T\,D_1^{-1}\,D_2^{-1} \ldots D_r^{-1}) = \dfrac{c^n\,\det A_1\,\det A_2 \ldots \det A_p\,\det B_1\,\det B_2 \ldots \det B_q}{\det D_1\,\det D_2 \ldots \det D_r}$

Cramers Rule

Cramers Rule can be used to find the unique solution x to $A\,x = b$ for a Square Matrix $A_{n\,n}$ with condition $\det A \neq 0$.

$x_i = \dfrac{\det A_i(b)}{\det A}$ *for each i where $1 \le i \le n$ and $A_i(b)$ is the matrix A with its column i replaced by the vector b*

Collecting together all x_p into a vector will produce the unique solution x to $A\,x = b$ for a Square Matrix $A_{n\,n}$.

Invertible Matrix Theorem

The Invertible Matrix Theorem is a list of equivalent statements. By being equivalent statements, if it is known or if it can be shown that any one of the statements is true, it follows that all of the statements are true. Conversely, if it is known or if it can be shown that any one of the statements is false, it follows that all of the statements are false.

Invertible Matrix Theorem

Let A be an n by n square matrix. All of the following statements are equivalent and either all are true or all are false.
If any single statement in the following list is known to be true, all the statements in the following list are true.
If any single statement in the following list is known to be false, all the statements in the following list are false.

a. A is an invertible matrix
   The matrix A is invertible if an inverse matrix A^{-1} exists as found by the augmented matrix method
b. The determinant of A is not zero $\det A \neq 0$
   The determinant of A is zero if the matrix contains at least one complete row or one complete column of zeros.
c. There exists a matrix C known as the inverse matrix of A such that $C\,A = A\,C = I$
   The matrix A is invertible if an inverse matrix $A^{-1} = C$ exists as found by the augmented matrix method
d. A^T is an invertible matrix
   Transpose matrix A^T is invertible if an inverse matrix $(A^T)^{-1}$ exists as found by the augmented matrix method
e. A is row equivalent to the n by n identity matrix I
   The matrix A can be row reduced through the reduction steps to become the n by n identity matrix I
f. A has n number of pivot positions
   The matrix A will row reduce to have a pivot position in every column for a total of n number of pivot positions
g. The columns of A span $\mathbb{R}^n$
   The matrix A will row reduce to have a pivot position in every column and n number of independent vectors
h. The columns of A form a linearly independent set
   The matrix A will row reduce to have a pivot position in every column and each of n columns is independent
i. The equation $A\,x = 0$ has only the trivial solution
   The matrix A will row reduce to the n by n identity matrix I and therefore will have the unique trivial solution
j. The equation $A\,x = b$ has a unique solution for each b in $\mathbb{R}^n$
   The matrix A will row reduce to the n by n identity matrix I and therefore will have a unique solution for each b
k. The linear transformation $x \mapsto A\,x$ is one to one
   The matrix A is one to one with a pivot position in every column and row reduces to the n by n identity matrix I
l. The linear transformation $x \mapsto A\,x$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
   The matrix A is one to one with a pivot position in every row and row reduces to the n by n identity matrix I
m. The columns of A form a basis of $\mathbb{R}^n$
   The matrix A will row reduce to have a pivot position in every column and n number of independent vectors
n. The span of the columns of A is $\mathbb{R}^n$ so that $Col\ A = \mathbb{R}^n$
   The matrix A will row reduce to have a pivot position in every column and n number of independent vectors
o. The dimension for the span of the columns of A is n so that $\dim Col\ A = n$
   The matrix A will row reduce to have a pivot position in every column and n number of independent vectors
p. The rank of A is the number of columns n of A so that $rank\ A = n$
   The matrix A will row reduce to have a pivot position in every column and n number of independent vectors
q. The nullspace of A contains only the zero vector so that $nul\ A = \{\vec{0}\}$
   The matrix A will row reduce to the n by n identity matrix I and therefore will have the unique trivial solution
r. The nullity of A is the number of free variables of A so that $nullity\ A = 0$
   The matrix A will row reduce to the n by n identity matrix I and therefore will have the unique trivial solution
s. 0 is not an eigenvalue of A
   The determinant of A is zero if the matrix contains at least one complete row or one complete column of zeros.
   By having at least one complete row or one complete column of zeros, the matrix will have 0 as at least one eigenvalue and therefore cannot possibly be a diagonalizable matrix

Vector Space and Vector Subspace

A Vector Space is a set of vectors under a set of axioms. A Vector Subspace is a closed space subset of a Vector Space.

Vector Space

A Vector Space is a vector set defined by the operations of vector addition and scalar multiplication in the ten axioms:

1. If any two vectors u and v are in the space, the addition vector $u + v$ must also be in the space.
2. The vector addition operation is commutative so that $u + v = v + u$ holds true.
3. The vector addition operation is associative so that $u + (v + w) = (u + v) + w$ holds true.
4. The space must contain the addition identity zero vector 0 so that $u + 0 = 0 + u = u$ holds true.
5. The space must contain the negative vector $-u$ so that $u + (-u) = (-u) + u = 0$ holds true.
6. If a vector v is in the space and scalar k is a real number, the multiplication $k\ v$ must also be in the space.
7. The scalar vector multiplication operation is distributive so that $k\ (u + v) = k\ v + k\ u$ holds true.
8. The scalar addition operation is distributive so that $(k + l)\ v = k\ v + l\ v$ holds true.
9. The scalar scalar vector multiplication operation is associative so that $k\ (l\ u) = (k\ l)\ u$ holds true.
10. The space must contain the multiplication identity one vector 1 so that $1\ u = u$ holds true.

Vector Subspace

A Vector Subspace is a vector space subset still closed under the operations of vector addition and scalar multiplication:

1. If any two vectors u and v are in the subspace, the addition vector $u + v$ must also be in the subspace.
2. If a vector v is in the subspace and scalar k is a real number, the multiplication $k\ v$ must also be in the subspace.
3. The subspace must contain the addition identity zero vector 0 so that $u + 0 = 0 + u = u$ holds true.

The first two conditions can be combined together by the statement if any two vectors u and v are in the subspace and scalars b and c are real numbers, the multiplication and addition vector $b\ u + c\ v$ must also be in the subspace.
The third condition requires that the zero vector 0 must always be present in the set for the possibility to be a subspace.

Form of a Subspace

The form of a Subspace requires linear conditions only that will allow for the existence of the zero vector 0 within the subspace. For these conditions to hold with the zero vector 0, the subspace definition should always have the form:

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{pmatrix} : \begin{matrix} a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 + \cdots + a_{1m}\ x_m = 0 \\ a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 + \cdots + a_{2m}\ x_m = 0 \\ a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 + \cdots + a_{3m}\ x_m = 0 \\ \vdots \\ a_{i1}\ x_1 + a_{i2}\ x_2 + a_{i3}\ x_3 + \cdots + a_{im}\ x_m = 0 \end{matrix} \right\}$$

The Subspace form definition equations on the right side have two necessary conditions that they must follow:

1. Each equation must be linear, with each x_j in the numerator raised to the first power only, multiplying a constant only and not each other, and not contained in any radicals, trig functions, exponentials, or logs.
2. Each equation must be homogenous, that is each equation should be equal only to zero on its right side.

Subspace, Span, and Coordinates

The first two Subspace identities ensure that the subspace is closed under vector addition and scalar multiplication. The third Subspace identity requires that the zero vector 0 must always be present in the set. Combining the three Subspace identities creates all possible linear combinations of the subspace vectors which is exactly the definition of the Span as

$$\beta = \left\{ x_1 \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ \vdots \\ c_{n1} \end{pmatrix} + x_2 \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \\ \vdots \\ c_{n2} \end{pmatrix} + x_3 \begin{pmatrix} c_{13} \\ c_{23} \\ c_{33} \\ \vdots \\ c_{n3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} c_{1q} \\ c_{2q} \\ c_{3q} \\ \vdots \\ c_{nq} \end{pmatrix} \right\} = span \left\{ \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ \vdots \\ c_{n1} \end{pmatrix}, \begin{pmatrix} c_{12} \\ c_{22} \\ c_{32} \\ \vdots \\ c_{n2} \end{pmatrix}, \begin{pmatrix} c_{13} \\ c_{23} \\ c_{33} \\ \vdots \\ c_{n3} \end{pmatrix}, \ldots, \begin{pmatrix} c_{1q} \\ c_{2q} \\ c_{3q} \\ \vdots \\ c_{nq} \end{pmatrix} \right\}$$

The subspace basis vectors c_j should be independent of each other and span the entire subspace with its conditions. Each ordered set of x_i values that solve a system $A\ x = b$ for any particular vector b can be expressed as a vector $[x]_\beta$ known as the coordinates for the vector b in the span of the subspace relative to the subspace basis vectors c_j.

Column Space

The Column Space is a Vector Subspace that can be formed as a linear combination from the columns of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The matrix can be written in the linear combination of vectors form, but some vectors may be dependent on the others.

$$Col\ A = x_1 \begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_2 \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_3 \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}$$

Form of Column Space

The Column Space is the set of columns formed by the linear combination of all Column Basis vectors or all independent column vectors in the original matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f represents a dependent column from the original matrix A and each column that contains a Pivot Position x_p represents an independent column from the original matrix A and the equivalent column from the original matrix A is one of the Column Space Basis Vectors. The remaining dependent columns can be ignored.

$$Col\ A = x_1 \begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_2 \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_3 \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}$$

Column Space Basis Vectors

The Basis Vectors of the Column Space is the collection of vectors d_j that are independent columns of a matrix A.

$$Basis\ Vectors = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}\ for\ Zero\ Matrix\ \ or\ \left\{ \begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix}, \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix}, \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix}, \ldots, \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix} \right\}\ for\ General\ Matrix$$

Column Space Dimension or Rank

The Rank is the Dimension of the Column Space and the number of Independent Column Vectors in the matrix. Once a matrix is in Echelon Form, each column that contains a Free Variable x_f is a dependent column and each column that contains a Pivot Position x_p is an independent column. The number of Pivot Positions x_p is the Rank of the matrix.

$$Rank\ A_{m\,n} + Nullity\ A_{m\,n} = Number\ of\ Pivot\ Positions + Number\ of\ Free\ Variables = n\ Number\ of\ Columns$$

Geometric Form for Span of Column Space Basis Vectors within $\mathbb{R}^m$ space

Number of Pivot Positions x_p in the Echelon Form matrix is the Rank or Dimension of the Column Space of the matrix.
If the Rank or the Dimension of the Column Space is zero, the Geometric Form of the Column Space is a Single Point.
If the Rank or the Dimension of the Column Space is one, the Geometric Form of the Column Space is a Line.
If the Rank or the Dimension of the Column Space is two, the Geometric Form of the Column Space is a Plane.
If the Rank or the Dimension of the Column Space is three, the Geometric Form of the Column Space is$3\ dim\ of\ \mathbb{R}^m$.
If the Rank or the Dimension of the Column Space is j, the Geometric Form of the Column Space is $j\ dim\ of\ \mathbb{R}^m$.
If the Rank or the Dimension of the Column Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Column Space is all of $\mathbb{R}^m$. This will occur if there is a Pivot Position x_p in every column.

Null Space

The Null Space a Vector Subspace that solves the homogenous linear system of equations in an augmented matrix

$$A\,x = 0$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & 0 \end{array}\right)$$

The Null Space or Homogenous Solution Space has all h_i equal to zero and the Vector Linear Combination Form is

$$Nul\ A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}$$

Form of Null Space

The form of the Null Space solution set is always consistent but depends on the number of solutions.

$$Nul\ A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad Exactly\ One\ Solution\ which\ is\ always\ the\ Trivial\ Solution$$

$$Nul\ A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = x_{f1}\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix} + x_{f2}\begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix} + x_{f3}\begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix} + \cdots + x_{fq}\begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix} \quad Infinite\ Number\ of\ Solutions$$

Null Space Basis Vectors

The Basis Vectors of the Null Space is the collection of vectors d_j multiplying with the Free Variables x_f.

$$Basis\ Vectors = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\}\ for\ One\ Solution \quad or \quad \left\{\begin{pmatrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{n1} \end{pmatrix}, \begin{pmatrix} d_{12} \\ d_{22} \\ d_{32} \\ \vdots \\ d_{n2} \end{pmatrix}, \begin{pmatrix} d_{13} \\ d_{23} \\ d_{33} \\ \vdots \\ d_{n3} \end{pmatrix}, \dots, \begin{pmatrix} d_{1q} \\ d_{2q} \\ d_{3q} \\ \vdots \\ d_{nq} \end{pmatrix}\right\}\ for\ Infinity\ Solutions$$

Null Space Dimension or Nullity

The Null Space Dimension or Nullity is the number of Basis Vectors q in the Linear Combination Form of a Null Space.
The Null Space Dimension or Nullity is the number of Free Variables x_f in the Reduced Echelon Form of the matrix.

$$Rank\ A_{m\,n} + Nullity\ A_{m\,n} = Number\ of\ Pivot\ Positions + Number\ of\ Free\ Variables = n\ Number\ of\ Columns$$

Geometric Form for Span of Null Space Basis Vectors

Number of Null Space Basis Vectors d_j is the Nullity or Dimension of the Solution Space of the matrix.
If the Nullity or the Dimension of the Null Space is zero, the Geometric Form of the Null Space is a Single Point.
If the Nullity or the Dimension of the Null Space is one, the Geometric Form of the Null Space is a Line.
If the Nullity or the Dimension of the Null Space is two, the Geometric Form of the Null Space is a Plane.
If the Nullity or the Dimension of the Null Space is three, the Geometric Form of the Null Space is $3\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Null Space is j, the Geometric Form of the Null Space is $j\ dim\ of\ \mathbb{R}^{\boldsymbol{n}}$.
If the Nullity or the Dimension of the Null Space is its maximum possible value of the columns in the matrix n, the Geometric Form of the Null Space is all of $\mathbb{R}^n$. This will occur if there is a Free Variable x_f in every column.

Eigenmatrix, Eigenvalues, Eigenvectors, and Diagonalization

An Eigenmatrix is a row dependent matrix created by an Eigenvalue λ and its corresponding Eigenvector v related by

$$A\,v = \lambda\,v$$

Where A is a given Square Matrix, which can be converted to its Eigenmatrix E to find the Eigenvalues and Eigenvectors.

Eigenmatrix

The Eigenmatrix E is the matrix A with the Eigenvalue λ subtracted from all diagonal terms of matrix A.

$$E = A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda \end{pmatrix}$$

Characteristic Polynomial

The Characteristic Polynomial $p(\lambda)$ of the variable Eigenvalue λ is formed by the determinant of the Eigenmatrix E.

$$p(\lambda) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda \end{pmatrix} = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda \end{vmatrix}$$

$$p(\lambda) = nth\ degree\ Characteristic\ Polynomial\ for\ an\ n\ by\ n\ Matrix\ A_{n\,n}$$

Eigenvalues

The Eigenvalues λ are the roots as the Characteristic Polynomial $p(\lambda)$ set equal to zero $p(\lambda) = 0$.

$$p(\lambda) = nth\ degree\ Characteristic\ Polynomial = 0 \qquad Solve\ for\ all\ n\ number\ of\ Eigenvalues\ \lambda$$

The Characteristic Polynomial can be completely factored through direct factorization or by the quadratic formula

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

The completely factored version of the Characteristic Polynomial will have the following form

$$p(\lambda) = (\lambda - b_1)^{k_1}\,(\lambda - b_2)^{k_2}\,(\lambda - b_3)^{k_3} \dots (\lambda - b_s)^{k_s}$$

Where each exponent k_p represents the multiplicity or number of root repetitions of its corresponding Eigenvalue b_p.

Eigenvectors

Each Eigenvalue λ has one or more Eigenvectors v associated with it as the homogenous solutions to the Eigenmatrix

$$E\,v = \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Plug in each Eigenvalue λ one at a time into the Eigenmatrix E which should make the Eigenmatrix E row dependent, its rows will be linear combinations or multiples of each other. Check that the Eigenmatrix E is row dependent for every Eigenvalue λ. Solve the homogenous solution $E\,v = 0$ to find all Eigenvectors v associated with each Eigenvalue λ.

Any specific Eigenvalue $\lambda = b$ can have at most k number of Eigenvectors $v_1, v_2, v_3, \dots, v_k$ associated with it though it may have fewer than k number of Eigenvectors $v_1, v_2, v_3, \dots, v_{less\ than\ k}$ where k is the multiplicity of the Eigenvalue $\lambda = b$, that is the number of repetitions of the root λ from the Characteristic Polynomial $p(\lambda)$ or the number of factors with the exact form $(\lambda - b)$ contained within the completely factorized version of the Characteristic Polynomial $p(\lambda)$.

Eigenspace

The set of Eigenvectors v for matrix A form a basis to nullspace $E\,v = 0$ of the Eigenmatrix E known as the Eigenspace.

Eigenvalues and Eigenvectors for a Linear System of Two Equations

A Linear System of Two Equations has the following standard form

$$a_{11}\,x_1 + a_{12}\,x_2 = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 = b_2$$

The solution to the system is produced by the two Eigenvalues λ_1 and λ_2 with their associated Eigenvectors v_1 and v_2.

1. Form the Matrix A and the Eigenmatrix E for the linear system of two equations

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad E = A - \lambda\, I = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

2. Form the Characteristic Polynomial $p(\lambda)$ by taking the determinant of the Eigenmatrix E and expanding.

$$p(\lambda) = \det\begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = (a_{11} - \lambda)\,(a_{22} - \lambda) - a_{12}\,a_{21}$$
$$p(\lambda) = \lambda^2 - (a_{11} + a_{22})\,\lambda + (a_{11}\,a_{22} - a_{12}\,a_{21}) = \lambda^2 - (trace\ A)\,\lambda + (\det A)$$

3. Set the Characteristic Polynomial $p(\lambda)$ equal to zero and factor or use the quadratic formula to find Eigenvalues.

$$p(\lambda) = 2nd\ degree\ Characteristic\ Polynomial = 0 \qquad Solve\ for\ 2\ Eigenvalues\ \lambda$$

4. Plug in each Eigenvalue $\lambda = b$ one at a time into the Eigenmatrix E. Check that the Eigenmatrix E is row dependent. Solve the homogenous solution $E\,v = 0$ to the Eigenvector v associated with each Eigenvalue λ.
For <u>Distinct Real Roots</u> $\lambda = \lambda_1$ and $\lambda = \lambda_2$ each root Eigenvalue will have an associated Eigenvector found by:

$$\begin{pmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a_{11} - \lambda_2 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$(a_{11} - \lambda_1)\,v_1 + a_{12}\,v_2 = 0 \qquad (a_{11} - \lambda_2)\,u_1 + a_{12}\,u_2 = 0$$
$$(a_{11} - \lambda_1)\,v_1 = -a_{12}\,v_2 = 0 \qquad (a_{11} - \lambda_2)\,u_1 = -a_{12}\,u_2$$
$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -a_{12} \\ a_{11} - \lambda_1 \end{pmatrix} \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -a_{12} \\ a_{11} - \lambda_2 \end{pmatrix}$$

In each case, factor the resulting Eigenvectors v and u down to its lowest positive integer values for simplicity.

Eigenvalues and Eigenvectors for a Linear System of Three Equations

A Linear System of Three Equations has the following standard form

$$a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 = b_2$$
$$a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 = b_3$$

The solution to the system is produced by the three Eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with their associated Eigenvectors v_1, v_2, v_3.

1. Form the Matrix A and the Eigenmatrix E for the linear system of three equations

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad E = A - \lambda\, I = \begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}$$

2. Form the Characteristic Polynomial $p(\lambda)$ by taking the determinant of the Eigenmatrix E and expanding.

$$p(\lambda) = \det\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix}$$

3. Set the Characteristic Polynomial $p(\lambda)$ equal to zero and factor or use the quadratic formula to find Eigenvalues.

$$p(\lambda) = 3rd\ degree\ Characteristic\ Polynomial = 0 \qquad Solve\ for\ 3\ Eigenvalues\ \lambda$$

4. Plug in each Eigenvalue $\lambda = b$ one at a time into the Eigenmatrix E. Check that the Eigenmatrix E is row dependent. Solve the homogenous solution $E\,v = 0$ to the Eigenvector v associated with each Eigenvalue λ.

For <u>Distinct Real Roots</u> $\lambda = \lambda_1$ and $\lambda = \lambda_2$each root Eigenvalue will have an associated Eigenvector found by:

$$\begin{pmatrix} a_{11} - \lambda_1 & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda_1 & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a_{11} - \lambda_2 & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda_2 & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In each case, factor the resulting Eigenvectors v and u down to its lowest positive integer values for simplicity.

Diagonalization

Diagonalization is the process by which a matrix A with certain necessary properties can be converted into a Diagonal Matrix D that has nonzero entries only on its diagonal and zero entries every else within the matrix. The Diagonal Matrix D will be sandwiched between a Place Matrix P and its corresponding Inverse Place Matrix P^{-1} in the following form:

$$A = P\,D\,P^{-1} \qquad \textit{which may also be expressed as} \qquad D = P^{-1}\,A\,P$$

Matrix A must have necessary conditions on its Eigenvalues λ and corresponding Eigenvectors v to be diagonalizable.

Diagonalization Matrix Forms

The Diagonal Matrix D is formed by all of the Eigenvalues λ of matrix A in any particular order but including exactly k repetitions for all multiplicities k of the Eigenvalues λ placed on its diagonal and zeros placed everywhere else.

$$D = \begin{pmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & d_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{mn} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & 0 & 0 & \dots & 0 \\ 0 & \lambda_{22} & 0 & \dots & 0 \\ 0 & 0 & \lambda_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{mn} \end{pmatrix} \textit{ where multiplicity } k \textit{ number of } \lambda \textit{ are equal}$$

The Place Matrix P and its corresponding Inverse Place Matrix P^{-1} are formed by all of the Eigenvectors v of matrix A in exactly the same order as the Eigenvalues λ in the Diagonal Matrix D but including a necessarily different Eigenvector for each multiplicity k of each Eigenvalues λ placed on the same columns as the Eigenvalues λ in Diagonal Matrix D.

$$P = (v_1|v_2|\dots|v_n) = \begin{pmatrix} v_{11} & v_{12} & v_{13} & \dots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \dots & v_{2n} \\ v_{31} & v_{32} & v_{33} & \dots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & v_{m3} & \dots & v_{mn} \end{pmatrix} \qquad P = (v_1|v_2|\dots|v_n)^{-1} = \begin{pmatrix} v_{11} & v_{12} & v_{13} & \dots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \dots & v_{2n} \\ v_{31} & v_{32} & v_{33} & \dots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & v_{m3} & \dots & v_{mn} \end{pmatrix}^{-1}$$

Necessary Conditions for Diagonalization

Matrix A must have necessary conditions on its Eigenvalues λ and corresponding Eigenvectors v to be diagonalizable.
The Eigenvalues λ are the roots as the Characteristic Polynomial $p(\lambda)$ set equal to zero $p(\lambda) = 0$.

$$p(\lambda) = nth\ degree\ Characteristic\ Polynomial = 0 \qquad Solve\ for\ all\ n\ number\ of\ Eigenvalues\ \lambda$$

The completely factored version of the Characteristic Polynomial will have the following form

$$p(\lambda) = (\lambda - b_1)^{k_1}\,(\lambda - b_2)^{k_2}\,(\lambda - b_3)^{k_3} \dots (\lambda - b_s)^{k_s}$$

Where each exponent k_p represents the multiplicity or number of root repetitions of its corresponding Eigenvalue b_p.
Each Eigenvalue λ has one or more Eigenvectors v associated with it as the homogenous solutions to the Eigenmatrix

$$E\,v = 0$$

A specific Eigenvalue $\lambda = b$ can have at most k number of Eigenvectors $v_1, v_2, v_3, \dots, v_k$ associated with it though it may have fewer than k number of Eigenvectors $v_1, v_2, v_3, \dots, v_{less\ than\ k}$ where k is the multiplicity of the Eigenvalue $\lambda = b$.

For a matrix A to be diagonalizable, it is necessary that each specific Eigenvalue $\lambda = b$ have exactly the maximum k number of Eigenvectors $v_1, v_2, v_3, \dots, v_k$ associated with it and not have any fewer than k number of Eigenvectors $v_1, v_2, v_3, \dots, v_{less\ than\ k}$ associated with it where k is the multiplicity of the Eigenvalue $\lambda = b$.

Applications of Diagonalization

Diagonalization allows for a diagonalizable matrix A to be raised to a large power k in much fewer calculations by

$$A^k = (P\,D\,P^{-1})^k = (P\,D\,P^{-1})(P\,D\,P^{-1})(P\,D\,P^{-1}) \dots (P\,D\,P^{-1}) = P\,D\,(P^{-1}P)\,D\,(P^{-1}P) \dots D\,(P^{-1}P)\,D\,P^{-1}$$

$$A^k = P\,D^k\,P^{-1}$$

The Diagonal Matrix D raised to a power k is equal to each of the diagonal entries λ themselves raised to the power k

$$A^k = P\,D^k\,P^{-1} = \begin{pmatrix} v_{11} & v_{12} & v_{13} & \dots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \dots & v_{2n} \\ v_{31} & v_{32} & v_{33} & \dots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & v_{m3} & \dots & v_{mn} \end{pmatrix} \begin{pmatrix} \lambda_{11}^k & 0 & 0 & \dots & 0 \\ 0 & \lambda_{22}^k & 0 & \dots & 0 \\ 0 & 0 & \lambda_{33}^k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{mn}^k \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} & v_{13} & \dots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \dots & v_{2n} \\ v_{31} & v_{32} & v_{33} & \dots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & v_{m3} & \dots & v_{mn} \end{pmatrix}^{-1}$$

Orthogonal Vectors, Orthonormal Vectors, and Basis

Orthogonal Vectors are a set of two or more vectors that are all mutually orthogonal to each other. Orthonormal Vectors are a set of two or more vectors that are all mutually orthogonal to each other and also are all normalized.

Euclidean Inner Product or Dot Product, Magnitude, and Normalized Unit Vector

The Euclidean Inner Product or Dot Product of two vectors $\vec{u}$ and $\vec{v}$ with equal number of components n results in the unscaled projection of each vector onto the other vector or the unscaled cosine of the vector angle and is defined as

$$\vec{u}\cdot\vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = u^T\, v = (u_1 \quad u_2 \quad u_3 \quad \dots \quad u_n) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = v^T\, u = (v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta = u_1\, v_1 + u_2\, v_2 + u_3\, v_3 + \cdots + u_n\, v_n$$

If two nonzero vectors $\vec{u}$ and $\vec{v}$ are mutually orthogonal, their Euclidean Inner Product or Dot Product will equal to zero

$$\vec{u}\cdot\vec{v} = 0 \quad or \quad \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2 \quad if\ and\ only\ if\ the\ nonzero\ vectors\ \vec{u}\ and\ \vec{v}\ are\ mutually\ orthogonal$$

Magnitude

The Magnitude $\|\vec{v}\|$ is the geometric length of a vector $\vec{v}$ in space and calculated through the Euclidean Inner Product

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}$$

Normalized Unit Vector

A Normalized Unit Vector $\vec{w}$ has a magnitude of exactly one but is in the same direction as some other vector $\vec{v}$.

$$\vec{w} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{\vec{v}\cdot\vec{v}}} \qquad so\ that\ vector\ \vec{w}\ has\ the\ direction\ of\ \vec{v}\ and\ a\ magnitude\ of\ \|\vec{w}\| = 1$$

A Vector $\vec{u}$ of magnitude k but in the direction of some other vector $\vec{v}$ can be calculated through a Unit Vector $\vec{w}$ as

$$\vec{u} = k\,\vec{w} = k\,\frac{\vec{v}}{\|\vec{v}\|} = k\,\frac{\vec{v}}{\sqrt{\vec{v}\cdot\vec{v}}} \qquad so\ that\ vector\ \vec{u}\ has\ the\ direction\ of\ \vec{v}\ and\ a\ magnitude\ of\ \|\vec{u}\| = k$$

Orthogonal Vector Set and Orthogonal Column Vector Matrix

An Orthogonal Vector Set is a collection of k number of vectors $v_1, v_2, v_3, \dots, v_k$ that are all mutually orthogonal to each other and therefore all Euclidean Inner Products or Dot Products will equal to zero for each possible ordered vector pair:

$$\{v_1, v_2, v_3, \dots, v_k\} \qquad is\ an\ orthogonal\ set\ if\ all\ v_i\cdot v_j = 0\ for\ all\ possible\ combinations\ of\ i, j\ where\ i \neq j$$

An Orthogonal Column Vector Matrix B has all n number of columns $v_1, v_2, v_3, \dots, v_n$ mutually orthogonal to each other and all column Euclidean Inner Products or Dot Products will equal to zero for each possible ordered column vector pair.

$$If\ Ax = b\ for\ an\ Orthogonal\ Matrix\ A, each\ solution\ variable\ is\ x_j = comp_{v_j}\, b = \frac{\vec{b}\cdot\vec{v_j}}{\vec{v_j}\cdot\vec{v_j}}\ for\ column\ \vec{v_j}$$

Orthonormal Set and Orthonormal Column Vector Matrix

An Orthonormal Vector Set is a collection of r number of vectors $v_1, v_2, v_3, \dots, v_r$ that are all mutually orthogonal to each other and therefore all Euclidean Inner Products or Dot Products will equal to zero for each possible ordered vector pair and each vector is a Normalized Unit Vector such that the magnitude of each vector is equal to exactly one:

$$\{v_1, v_2, v_3, \dots, v_k\} \qquad is\ an\ orthonormal\ set\ if\ all\ v_i\cdot v_j = 0\ for\ all\ possible\ combinations\ of\ i, j\ where\ i \neq j$$
$$and\ each\ vector\ \vec{v}_k\ is\ a\ unit\ vector\ such\ that\ all\ \|\vec{v}_k\| = 1\ for\ all\ possible\ values\ of\ k$$

An Orthonormal Column Vector Matrix B has all n number of columns $v_1, v_2, v_3, \dots, v_n$ mutually orthogonal to each other and each column vector is a Normalized Unit Vector. An Orthonormal Column Vector Matrix B has the properties:

$$B^T\, B = B\, B^T = I \qquad (B\,\vec{u})\cdot(B\,\vec{v}) = \vec{u}\cdot\vec{v}$$
$$\|B\,\vec{v}\| = \|\vec{v}\| \qquad proj_w\,\vec{v} = B\,B^T\,\vec{v}$$

$$If\ Ax = b\ for\ an\ Orthonormal\ Matrix\ A, each\ solution\ variable\ is\ x_j = comp_{v_j}\, b = \frac{\vec{b}\cdot\vec{v_j}}{\vec{v_j}\cdot\vec{v_j}} = \vec{b}\cdot\vec{v_j}\ for\ column\ \vec{v_j}$$

Orthogonal Projection

The Orthogonal Projection $\hat{y} = proj_v\, y$ of a vector $\vec{y}$ onto another vector $\vec{v}$ is the shadow component vector of the first vector $\vec{y}$ in the span of the vector $\vec{v}$ or parallel with the vector $\vec{v}$ and $z = y - \hat{y} = y - proj_v\, y$ is the perpendicular component vector of the first vector $\vec{y}$ to the span of the vector $\vec{v}$ as follows:

$$\hat{y} = proj_v\, y = \left(\frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} \quad \textit{component vector of } \vec{y} \textit{ parallel to } \vec{v} \textit{ or in span } \{\vec{v}\} \textit{ where each} \quad comp_v\, y = \left(\frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)$$

$$z = y - \hat{y} = y - proj_v\, y = \vec{y} - \left(\frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} \quad \textit{component vector of } \vec{y} \textit{ perpendicular to } \vec{v} \textit{ or perp to span } \{\vec{v}\}$$

The Orthogonal Projection $\hat{y} = proj_w\, y$ of a vector $\vec{y}$ onto a vector set $w = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}\}$ is the shadow component vector of the first vector $\vec{y}$ in the span of the vector set $w = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}\}$ and $z = y - \hat{y} = y - proj_w\, y$ is the perpendicular component vector of the first vector $\vec{y}$ to the span of the vector set $w = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}\}$ as follows:

$$\hat{y} = proj_w\, y = \left(\frac{\vec{y}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\right)\overrightarrow{v_1} + \left(\frac{\vec{y}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\right)\overrightarrow{v_2} + \cdots + \left(\frac{\vec{y}\cdot\overrightarrow{v_r}}{\overrightarrow{v_r}\cdot\overrightarrow{v_r}}\right)\overrightarrow{v_r} \quad \textit{in span } \{w\} \textit{ where each} \quad comp_v\, y = \left(\frac{\vec{y}\cdot\overrightarrow{v_r}}{\overrightarrow{v_r}\cdot\overrightarrow{v_r}}\right)$$

$$z = y - \hat{y} = y - proj_w\, y = \vec{y} - \left(\frac{\vec{y}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \vec{y} - \left(\frac{\vec{y}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\right)\overrightarrow{v_1} - \left(\frac{\vec{y}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\right)\overrightarrow{v_2} - \cdots - \left(\frac{\vec{y}\cdot\overrightarrow{v_r}}{\overrightarrow{v_r}\cdot\overrightarrow{v_r}}\right)\overrightarrow{v_r} \quad \textit{perp to span } \{w\}$$

The Gram-Schmidt Process to form an Orthogonal Basis or an Orthonormal Basis

The Gram-Schmidt Process is a method that will convert a general basis vector set $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}, \dots, \overrightarrow{x_r}\}$ of subspace W into an equivalent orthogonal basis vector set $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}\}$ also of subspace W. The two vector sets will be a basis to the same subspace W and will therefore have identical spans $W = span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}\} = span\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}, \dots, \overrightarrow{x_r}\}$.

$$\overrightarrow{v_1} = \overrightarrow{x_1}$$

$$\overrightarrow{v_2} = \overrightarrow{x_2} - \left(\frac{\overrightarrow{x_2}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\right)\overrightarrow{v_1}$$

$$\overrightarrow{v_3} = \overrightarrow{x_3} - \left(\frac{\overrightarrow{x_3}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\right)\overrightarrow{v_1} - \left(\frac{\overrightarrow{x_3}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\right)\overrightarrow{v_2}$$

$$\vdots$$

$$\overrightarrow{v_r} = \overrightarrow{x_r} - \left(\frac{\overrightarrow{x_r}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\right)\overrightarrow{v_1} - \left(\frac{\overrightarrow{x_r}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\right)\overrightarrow{v_2} - \left(\frac{\overrightarrow{x_r}\cdot\overrightarrow{v_3}}{\overrightarrow{v_3}\cdot\overrightarrow{v_3}}\right)\overrightarrow{v_3} - \cdots - \left(\frac{\overrightarrow{x_r}\cdot\overrightarrow{v_r}}{\overrightarrow{v_r}\cdot\overrightarrow{v_r}}\right)\overrightarrow{v_r}$$

The Orthogonal Basis vector set can be formed as $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}\}$ and it also spans the same subspace W as the basis vector set $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}, \dots, \overrightarrow{x_r}\}$ but the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_r}$ are all mutually orthogonal to each other.

The Orthonormal Basis vector set can be formed as $\left\{\frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|}, \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|}, \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|}, \dots, \frac{\overrightarrow{v_r}}{\|\overrightarrow{v_r}\|}\right\}$ and it also spans the same subspace W as the basis vector set $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}, \dots, \overrightarrow{x_r}\}$ but the vectors $\frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|}, \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|}, \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|}, \dots, \frac{\overrightarrow{v_r}}{\|\overrightarrow{v_r}\|}$ are all mutually orthogonal to each other and the vectors $\frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|}, \frac{\overrightarrow{v_2}}{\|\overrightarrow{v_2}\|}, \frac{\overrightarrow{v_3}}{\|\overrightarrow{v_3}\|}, \dots, \frac{\overrightarrow{v_r}}{\|\overrightarrow{v_r}\|}$ are all Normalized Unit Vectors and have magnitudes of exactly one.

Least Squares Solution Method

The Least Squares Solution Method determines the best fit solutions x to the inconsistent linear equation set $A\,x = b$.

$$(A^T\,A)\,\hat{x} = (A^T\,b) \quad \textit{has the least square solutions } \hat{x} \textit{ if } A\,x = b \textit{ is an inconsistent linear equation set}$$

The Least Squares Solution Method finds the best fit solutions $\hat{x}$ but has an error of $Least\ Squares\ Error = \|b - A\,\hat{x}\|$

QR Factorization

QR Factorization is a method that factors a general m by n matrix $A = Q\,R$ into a multiplication of an m by n matrix Q whose columns form an equivalent orthonormal basis of $col\ A$ and an n by n square matrix R that is upper triangular by:

1. Perform the Gram-Schmidt Process on all columns of matrix A and put in their same positions to form matrix Q.
2. Calculate the upper triangular square matrix R by the equation $R = Q^T\,A$.

Symmetric Matrix

A Symmetric Matrix A is a square matrix that has a mirror image form between its upper triangular portion above the diagonal and its lower triangular portion below the diagonal such that the matrix A is equal to its own transpose A^T.

$$A = A^T \qquad \textit{If the matrix A is a symmetric matrix}$$

Expanding the form of the matrix A and its transpose matrix A^T leads to the relation between the matrix components.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix}$$

The relation between the components of matrix A and transpose matrix A^T requires the symmetric matrix to have form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix} \qquad \textit{is the form if the matrix A is a symmetric matrix}$$

Symmetric Matrix Properties

Symmetric Matrix A properties for Eigenvalues λ, Eigenvectors v, Eigenspace $E\ v = 0$, Diagonal Matrix D, Place Matrix P

Symmetric Matrix Eigenvalues will always be real number values for all n number of Eigenvalues.
The Eigenvalues λ are the roots as the Characteristic Polynomial $p(\lambda)$ set equal to zero $p(\lambda) = 0$.

Symmetric Matrix Eigenvectors v are orthogonal to each other and will always form a mutually orthogonal set.
Each Eigenvalue λ has one or more associated Eigenvectors v as homogenous solutions $E\ v = 0$ to the Eigenmatrix.

Symmetric Matrix Eigenspace dimension for each Eigenvalue λ will equal exactly the multiplicity of the Eigenvalue.
The set of Eigenvectors v for matrix A form a basis to nullspace $E\ v = 0$ of the Eigenmatrix E known as the Eigenspace.

Symmetric Matrix Diagonal Matrix D always exists and the Symmetric Matrix is therefore diagonalizable.
The Diagonal Matrix D is formed by all of the Eigenvalues λ of matrix A in any particular order but including exactly k repetitions for all multiplicities k of the Eigenvalues λ placed on its diagonal and zeros placed everywhere else.

Symmetric Matrix Place Matrix P is orthogonal and the Symmetric Matrix is orthogonally diagonalizable.
The Place Matrix P is formed by all of the Eigenvectors v of matrix A in exactly the same order as the Eigenvalues λ in the Diagonal Matrix D but including a necessarily different Eigenvector for each multiplicity k of each Eigenvalues λ.

Quadratic Forms

Quadratic Forms are a linear combination of squared variables polynomials that can be expressed as a Symmetric Matrix

$$\textit{Quadratic Form} = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 + \cdots + a_{ij}x_ix_j$$

$$\textit{Quadratic Form Symmetric Matrix} = x^T\ A\ x = (x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n) \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} & \cdots & \frac{1}{2}a_{2n} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} & \cdots & \frac{1}{2}a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \frac{1}{2}a_{3n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

Change of Variables to Remove Cross Product Terms x_ix_j from Matrix A is an orthonormal diagonalization. Change of variables $x = P\ y$ with a normalized Place Matrix P leads to the Quadratic Matrix Form $x^T\ D\ x$ with Diagonal Matrix D.