GradePeak
private instruction

BUSINESS CALC

CONCEPT PACKET

Complete

Written by Jeremy Robinson, Head Instructor

Find Out More
+Private Instruction
+Review Sessions

WWW.GRADEPEAK.COM

Need Help?

Online Private Instruction
Anytime, Anywhere
Any High School!
Any College!

Individual or Small Group

Visit www.gradepeak.com

Call or Text 814.867.1777

Important Derivative and Antiderivative Formulas

Derivative Formulas where u is a function of x

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(k\,x) = k$

$\frac{d}{dx}(u^n) = n\,u^{n-1}\frac{du}{dx}$

Derivative Identities where u and v are functions of x

$\frac{d}{dx}(k\,u) = k\frac{du}{dx}$

$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

$\frac{d}{dx}(u \times v) = \frac{du}{dx}v + u\frac{dv}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

$(k\,u)' = k\,u'$

$(u+v)' = u' + v'$

$(u-v)' = u' - v'$

$(u \times v)' = u'\,v + u\,v'$

$\left(\frac{u}{v}\right)' = \frac{u'\,v - u\,v'}{v^2}$

Definition of the Derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Antiderivative Formulas where u is a function of x

$$\int k\,du = k\,u + C$$

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

Antiderivative Properties

$$\int c\,f(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Important Formulas

Pythagorean Theorem

$a^2 + b^2 = c^2$

Quadratic Equation

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Economics Functions

x is the number of units demanded or sold

$p(x)$ is the Demand Equation or the Price Equation, relating the price demanded to the number of units sold

$R(x) = x\,p(x)$ is the Revenue Function, relating the revenue generated to the number of units sold

$C(x)$ is the Cost Function, relating the cost generated to the number of units sold

$P(x) = R(x) - C(x)$ is the Profit Function, relating the profit generated to the number of units sold

Average Functions

$\overline{R(x)} = \frac{R(x)}{x}$ is the Average Revenue Function, or the average revenue generated per unit sold

$\overline{C(x)} = \frac{C(x)}{x}$ is the Average Cost Function, or the average cost generated per unit sold

$\overline{P(x)} = \frac{P(x)}{x}$ is the Average Profit Function, or the average profit generated per unit sold

Marginal Functions

$R'(x)$ is the Marginal Revenue Function, or the derivative of the Revenue Function

$C'(x)$ is the Marginal Cost Function, or the derivative of the Cost Function

$P'(x)$ is the Marginal Profit Function, or the derivative of the Profit Function

Marginal Average Functions

$\overline{R'(x)}$ is the Marginal Average Revenue Function, or the derivative of the Average Revenue Function

$\overline{C'(x)}$ is the Marginal Average Cost Function, or the derivative of the Average Cost Function

$\overline{P'(x)}$ is the Marginal Average Profit Function, or the derivative of the Average Revenue Function

Function Domain and Function Value

The Function Domain and Function Value both involve the evaluation of a function with an independent variable.

Function Domain

The Function Domain is the set or collection of all x values that can be plugged into a function.
The Function Domain is All Real Numbers $(-\infty, +\infty)$ except in the following cases:

1. The function has a denominator, the denominator set as does not equal zero will give the Domain
   If $f(x) = \frac{g(x)}{h(x)}$ then the condition $h(x) \neq 0$ will give the Function Domain
2. The function has an even root, the inside part set greater than or equal to zero will give the Domain
   If $f(x) = \sqrt[n]{g(x)} = (g(x))^{\frac{1}{n}}$ with n even, then the condition $g(x) \geq 0$ will give the Function Domain

The Function Domain can be written in inequality notation, but it is more commonly written in bracket notation.
Square brackets $[\ ]$ are used for values that are on the end or extreme of an interval and are included in the interval.
Round brackets $(\ )$ are used for values that are on the end or extreme of an interval but are not included in the interval.
Positive and Negative Infinity extreme values of an interval are never included in the interval and have round brackets.

Function Value

The Function Value $f(a)$ is the evaluation of the y coordinate value of a function when its x coordinate value exactly equals a specific number a. If a Function Value exists, there is a filled in point at the number a itself.
The Function Value $f(a)$ exists at all values of a except where a produces one of the following results:

1. Zero in the denominator; the Function Value $f(a)$ in this case Does Not Exist.
2. Negative Number in an even root; the Function Value $f(a)$ in this case Does Not Exist.

Function Limit and Function Continuity

The Function Limit is a very different concept than the Function Value. In a Function Value, we are evaluating the y coordinate value of a function when its x coordinate value is exactly a specific number a. In a Limit we are evaluating the y coordinate value of a function as its x coordinate value approaches a specific number a but while the x coordinate value is very close to the number a, it never actually equals the number a itself.

One Sided Limit

For the One Sided Limit, we are investigating the y coordinate value of a function as its x coordinate value approaches a specific number a from just one side, either the side where the x is close to but always less than a (known as the Left Side Limit) or the side where x is close to but always greater than a (known as the Right Side Limit).
Left Side Limit Notation $\lim_{x \to a^-} f(x)$ x is close to but always less than a
Right Side Limit Notation $\lim_{x \to a^+} f(x)$ x is close to but always greater than a
If these two One Sided Limit results match each other, the Two Sided Limit is also this same value.

Two Sided Limit

For the Two Sided Limit, we are investigating the y coordinate value of a function as its x coordinate value approaches a specific number a from both sides, where the x is close to a but both less than a and greater than a.
Two Sided Limit Notation $\lim_{x \to a} f(x)$ x is close to a but both less than a and greater than a.
There are two possible results for the Two Sided Limit, depending on the results of both of the One Sided Limits:

1. If the two One Sided Limit results match each other, the Two Sided Limit is also this same value.
   If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ then we can also state that $\lim_{x \to a} f(x) = L$
   In this situation the graph does link up from both sides, but this does not verify a point there and leaves a hole.
2. If the two One Sided Limit results are different, the Two Sided Limit Does Not Exist (DNE).
   If $\lim_{x \to a^-} f(x) = L$ but $\lim_{x \to a^+} f(x) = M$ then we can also state that $\lim_{x \to a} f(x) = DNE$
   In this situation the graph does not link up from both sides, but instead has a jump discontinuity.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Breaking a Two Sided Limit into two One Sided Limits

There are several cases where it is necessary to break a Two Sided Limit into two One Sided Limits. These include:

1. The function is a piecewise function, defined with different equations on different domains and the limit involves the exact x coordinate where the function changes from one equation to another. Different equations are often needed in each one of the One Sided Limits, but still may produce the same result.
2. The function contains a denominator, and the limit involves the exact x coordinate that makes just the expression in the denominator equal to zero. These limits result in either positive or negative infinity. Negative exponent functions, as well as $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ are all denominator functions.
3. The function contains an even root, and the limit involves the exact x coordinate that makes the expression inside of the even root equal to zero. Exponents with an even number in the denominator of the exponent itself are actually even root functions. This limit equals to zero from one side, but Does Not Exist from the other side.
4. The function has both a numerator and a denominator, one which contains an absolute value, and the limit involves the exact x coordinate that makes the expression inside of the absolute value equal to zero.

Evaluating Limits

For many Function Limits, it is possible to evaluate them in the same way as evaluating a Function Value by simply plugging in the value a that the x coordinate is approaching. There are several exceptions when this is not possible:

1. The function contains a numerator and a denominator, and the limit involves the exact x coordinate that makes both the expression in the numerator and the expression in the denominator equal to zero. The Function Limit will have a form of zero divided by zero, an Indeterminate Form that must be reworked with algebra.
$$\lim_{x\to a} f(x) = \frac{0}{0} \qquad \textit{Indeterminate}$$
The value of this limit result cannot be determined immediately and it is necessary to rework the problem using algebra until a cancellation of terms occurs such that the value of the Function Limit is no longer Indeterminate. The algebra method to achieve this depends on the function:
    a. For power functions in either the numerator or the denominator, expand any expression powers, combine terms, factor the result, and cancel any like terms in the numerator and denominator
    b. For fraction functions in either the numerator or the denominator, find a common denominator first and combine the fractions together, then reciprocate and multiply to turn the overall expression into a single fraction. Proceed to Step a to complete the limit.
    c. For square root functions added or subtracted with power functions, multiply both the numerator and denominator by the root conjugate of the expression. Proceed to Step a to complete the limit.
    d. For trig functions, convert everything into sines and cosines and use the sine limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$
2. The function contains a numerator and a denominator, and the limit involves the exact x coordinate that makes both the expression in the numerator and the expression in the denominator equal to infinity. The Function Limit will have a form of infinity divided by infinity, an Indeterminate Form that must be reworked with algebra.
$$\lim_{x\to a} f(x) = \frac{\infty}{\infty} \qquad \textit{Indeterminate}$$
Divide the numerator and denominator by the highest order function in the denominator and retry the limit.
3. The function contains a denominator, and the limit involves the exact x coordinate that makes the expression in the denominator equal to zero. These limits result in either positive or negative infinity. For a Two Sided Limit, it is necessary to break it into two One Sided Limits. The type of infinity that each side produces is determined by plugging in the a and evaluating the overall sign of the expression. These are known as vertical asymptotes.
$\lim_{x\to a} f(x) = \frac{c}{0} = \pm\infty \quad \textit{where } c \neq 0$ and the denominator is a positive or negative small number
4. The function contains an even root or an exponent whose denominator is even, and the limit involves the exact x coordinate that makes the expression in the root equal to zero. For a Two Sided Limit, it is necessary to break it into two One Sided Limits. Those that result in an even root of a positive zero will be equal to zero, and those that result in an even root of a positive zero will be Does Not Exist.
$$\lim_{x\to a} f(x) = \sqrt[n]{u} = 0 \quad \textit{if n is even and } u = 0^+$$
$$\lim_{x\to a} f(x) = \sqrt[n]{u} = \textit{Does Not Exist} \quad \textit{if n is even and } u = 0^-$$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Limit Results and Indeterminate Forms

$+\infty$ indicates a boundless large positive number and $-\infty$ indicates a boundless large negative
$+0$ indicates a very small positive number near 0 and -0 indicates a very small negative number near 0.
k represents a finite nonzero positive constant $k > 0$.

Results involving a small positive number $+0$ or a small negative number -0

$+0/k$ =small positive number/positive constant=small positive number=$+0$=0
$-0/k$=small negative number/positive constant=small negative number=-0=0
$k/+0$=positive constant/small positive number=large positive number=$+\infty$
$k/-0$=positive constant/small negative number=large negative number=$-\infty$
$\sqrt{+0}$=square root of a small positive number=small positive number=$+0$=0
$\sqrt{-0}$=square root of a small negative number=DNE or Does Not Exist: negative number not in the domain
$\ln(+0)$ = natural log of a small positive number=large negative number=$-\infty$
$\ln(-0)$ = natural log of a small negative number= DNE or Does Not Exist: negative number not in the domain

Results involving a large positive number $+\infty$ or a large negative number $-\infty$

$+\infty/k$ =large positive number/positive constant=large positive number=$+\infty$
$-\infty/k$ =large negative number/positive constant=large negative number=$-\infty$
$k/+\infty$ =positive constant/large positive number=small positive number=$+0$=0
$k/-\infty$=positive constant/large negative number=small negative number-0=0
$k^{+\infty}$=positive constant raised to a large positive number=$+\infty$ if k>1 and $+0$=0 if 0<k<1
$k^{-\infty}$=positive constant raised to a large negative number=small positive number=$+0$=0
$\sqrt{+\infty}$=square root of a large positive number=large positive number=$+\infty$
$\ln(+\infty)$=natural log of a large positive number=large positive number=$+\infty$

Results involving combinations of $+\infty$, $-\infty$, $+0$, and -0

$\pm 0/\pm\infty$=small positive or negative number/large positive or negative number=small positive number=$+0$=0
$+0^{+\infty}$=small positive number raised to a large positive number=small positive number=$+0$=0
$+0^{-\infty}$=small positive number raised to a large negative number=large positive number=$+\infty$
$\pm 0/\pm 0$=small positive or negative number/small positive or negative number=$Indeterminate$
$\pm\infty/\pm\infty$=large positive or negative number/large positive or negative number=$Indeterminate$
$\pm 0 * \pm\infty$=small positive or negative number*large negative or positive number=$Indeterminate$
$\infty - \infty$=large positive number-large positive number=$Indeterminate$
$\pm 0^{\pm 0}$=small positive or negative number raised to small positive or negative number=$Indeterminate$
$\pm\infty^{\pm 0}$=large positive or negative number raised to small positive or negative number=$Indeterminate$
$1^{\pm\infty}$=number close to 1 but greater than 1 raised to a large positive or negative number=$Indeterminate$
$1^{\pm\infty}$=number close to 1 but lesser than 1 raised to a large positive or negative number=$Indeterminate$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

L'Hospitals Rule

L'Hospitals Rule can be used only when the limit results in either one of the two $L'Hospitals\ Indeterminate\ Forms$

$$L'Hospitals\ Indeterminate\ Forms \qquad \frac{\pm 0}{\pm 0} \qquad \frac{\pm\infty}{\pm\infty}$$

If a limit results in either of the two $L'Hospitals\ Indeterminate\ Forms$, the limit can be calculated following the steps:

1. Verify the limit results in one of the two $L'Hospitals\ Indeterminate\ Forms$.
2. Separately calculate the derivative of the numerator and denominator of the expression and retry the limit:
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$
3. Repeat the derivative of the numerator and denominator of the expression and retry the limit until the result is no longer a $L'Hospitals\ Indeterminate\ Form$ and instead will be the final result of the limit.

It is possible to convert most other $Indeterminate$ forms into a L'Hospitals Form including:

$$Convertible\ Indeterminate\ Forms \qquad \pm 0 * \pm\infty \qquad \infty - \infty \qquad \pm 0^{\pm 0} \qquad \pm\infty^{\pm 0} \qquad 1^{\pm\infty}$$

This can be achieved by the following processes depending on the type of $Convertible\ Indeterminate\ Form$

$\pm 0 * \pm\infty$ For results of this type take either one of the two function terms that are being multiplied, reciprocate it and place it into the denominator. This will not change the value of the expression, only its form. The form should now be one of the $L'Hospitals\ Indeterminate\ Forms$ depending on which one of the two terms is reciprocated.

$\infty - \infty$ For results of this type, several possibilities exist to change its form depending on the two subtracting terms:
1) If either of the two subtracting expressions has a denominator, find a common denominator. The form should now be one of the $L'Hospitals\ Indeterminate\ Forms$.
2)If one or both of the two subtracting terms are square roots, the form can be changed by multiplying the expression by their root conjugate over itself. The root conjugate is the same as the original expression with a plus sign between the two terms instead of a minus sign. The form should now be one of the $L'Hospitals\ Indeterminate\ Forms$.
3)If the terms have a common factor, factor it. The form should now be one of the $L'Hospitals\ Indeterminate\ Forms$.

$\pm 0^{\pm 0}, \pm\infty^{\pm 0}, 1^{\pm\infty}$ For results of this type, raise the elementary base e to the natural log of the expression in the limit and bring the exponent out to the front of the natural log. The form will become $\pm 0 * \pm\infty$ which has a method above. The answer is always of the form e^{number} so that $e^0 = 1$, $e^{\ln b} = b$, $e^{-\infty} = 0$, and $e^{+\infty} = +\infty$ are all possible answers.

Leading Term Analysis

Leading Term Analysis is used for **infinite limits** where L'Hospital's Rule fails. Leading Term Analysis involves calculating the limit after dividing both the numerator and denominator of the fraction function by the leading term in the denominator or simply after taking the ratio of the individual leading terms from the numerator and denominator.

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{leading\ term\ in\ f(x)}{leading\ term\ in\ g(x)} = \begin{cases} zero \quad 0 & if\ leading\ term\ in\ f(x) < leading\ term\ in\ g(x) \\ coefficient\ ratio & if\ leading\ term\ in\ f(x) = leading\ term\ in\ g(x) \\ infinite \quad \pm\infty & if\ leading\ term\ in\ f(x) > leading\ term\ in\ g(x) \end{cases}$$

Leading Terms from greatest to least for $x \to \infty$

1) Power Exponential $x^x, (kx)^x$ which can be subordered by the value of the multiplier k. For example $(2x)^x < (3x)^x$
2) Quadratic Exponential a^{x^2} which can be subordered by the value of the base a. For example $2^{x^2} < e^{x^2} < 3^{x^2}$
3) Factorial $x!, (kx)!$ which can be subordered by the value of the multiplier k. For example $x! < (2x)! < (3x)!$
4) Linear Exponential a^x which can be subordered by the value of the base a. For example $2^x < e^x < 3^x < 4^x$
5) Power x^n which can be subordered by the value of the power p. For example $x^{\frac{1}{2}} < x < x^{\frac{3}{2}} < x^2 < x^{\frac{5}{2}} < x^3$
6) Logarithm $\log_a x, \ln x$ which are all of the same order regardless of the value of the base a.
7) Bounded Functions $constant\ C,\ (-1)^x,\ C + (-1)^x$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Continuity

When a function is called continuous, it implies that the function is continuous for each and every point in its domain
When a function is called discontinuous, it implies that the function is not continuous for at least one point in its domain
The continuity of a function $f(x)$ at a single point $x = a$ is defined as

$$\lim_{x\to a} f(x) = f(a)$$

Remember that in order for a Two Sided Limit to exist, the two One Sided Limits must themselves have an equal value.
There are several types of continuity and discontinuity:

1. If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$ the function is Continuous at the point $x = a$. The Function Limit will link the graph together from both sides, with the empty hole filled in by the Function Value.
2. If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$ the function has a Removable Discontinuity at the point $x = a$. The Function Limit will link the graph together from both sides, but there exists an empty hole at the link from the Function Value being different. There may still be a point above or below the hole.
3. If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ the function has a Jump Discontinuity at the point $x = a$. The Function Limit will not link the graph together from both sides and the graph will jump at this point.

Limits at Infinity

Limits can also be taken as x approaches positive or negative infinity. Things to watch for in evaluating any infinite limit:

1. If the given function is fraction where both the numerator and the denominator are polynomials, try dividing both the numerator and denominator by the highest order (or power) of x. This will create a fraction equation where many of the terms will be $\frac{c}{x^n}$, and these terms will become zero as x approaches infinity since a constant divided by a very large number is zero. The limit can then be evaluated.
2. If the given function is an addition or subtraction involving a square root, it can be evaluated by multiplying the numerator and denominator by the root conjugate to turn the function into a fraction expression.
3. If the problem involves a single root but no addition or subtraction, it is possible to evaluate the limit inside the root first and then take the root of this limit for the final answer.

A Limit at positive infinity is the horizontal asymptote of a function on the far right side of its graph.
A Limit at negative infinity is the horizontal asymptote of a function on the far left side of its graph

Derivatives

The derivative is the foundation to all of Calculus, and is defined as the slope of a tangent line to a function graph.

Definition of a Derivative

The derivative is the slope of a line tangent to a curve. A tangent line is the limit of secant lines drawn between two points a distance apart. By taking the limit as this distance goes to zero, the points through which the secant lines passes become closer together until eventually the secant line becomes a tangent line. This is definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

The Derivative Function Limit will always result in a form of zero divided by zero, an Indeterminate Form that must be reworked with algebra. The goal is to be able to cancel out either the h or the $x - a$ term from the denominator.

$$\lim_{x\to a} f(x) = \frac{0}{0} \qquad \textit{Indeterminate}$$

The algebra method to cancel the denominator term and find the Definition of the Derivative depends on the function:

a. For power functions in either the numerator or the denominator, expand any expression powers, combine terms, factor the result, and cancel any like terms in the numerator and denominator
b. For fraction functions in either the numerator or the denominator, find a common denominator first and combine the fractions together, then reciprocate and multiply to turn the overall expression into a single fraction. Proceed to Step a to complete the limit.
c. For square root functions added or subtracted with power functions, multiply both the numerator and denominator by the root conjugate of the expression. Proceed to Step a to complete the limit.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Derivative Identities

The Derivative Identities can be used to determine the derivative of two or more functions combined together.

Sum Difference Rule The derivative for any number of functions that are either added together or subtracted together

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} = u' \pm v'$$

Product Rule The derivative for any two functions multiplied together

$$\frac{d}{dx}(u\,v) = \frac{du}{dx}\,v + u\,\frac{dv}{dx} = u'\,v + u\,v'$$

Quotient Rule The derivative for any two functions divided together

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\,v - u\frac{dv}{dx}}{v^2} = \frac{u'\,v - u\,v'}{v^2}$$

Chain Rule The derivative for any two or more functions together as a composite function. The Chain Rule is always done in the exact opposite order of the operations involved when plugging in a value for x. The derivative of the outermost function is done first using its formula while retaining the value of all inner functions, then multiplying this result by the derivative of the next inner function which may require another Chain Rule.

$$\big(f(u)\big)' = f'(u)\,u' \qquad\qquad \Big(g\big(f(u)\big)\Big)' = g'\big(f(u)\big)\,f'(u)\,u'$$

Function Differentiability

A function is differentiable at a point $x = a$ if it has a derivative or a single slope of a tangent line defined at that point. The definition of the derivative involves a limit and therefore requires:

In order for a function to be differentiable at a point $x = a$, it must first be continuous at that same point $x = a$.

However, a function can be continuous and yet not be differentiable. This happens for the functions that have a graph with a Line Point such as the absolute value function, or functions that have a Cusp Point such as certain fraction exponential functions. It is just necessary that the function is continuous before any discussion of its differentiability.

Common functions that may not be differentiable:

1. If the function contains a denominator, the function is not differentiable at the exact x coordinate that makes the expression in the denominator equal to zero.
2. If the function contains an even root or an exponent whose denominator is even, the function is not differentiable at the exact x coordinate that makes the expression in the root equal to zero. This value is actually an endpoint to the graph, and endpoints are never differentiable.
3. If the function is a piecewise function, it may or may not be differentiable at the location where the function changes from one equation expression into another equation expression. To check the differentiability, take the derivative each piecewise function and check the continuity of this new piecewise function through the use of the Function Limit. If the result of the derivative is continuous the original function is differentiable. If the result of the derivative is not continuous, the original function is not differentiable.
4. If a function contains a removable discontinuity or a jump discontinuity, it is not differentiable at the exact x coordinate where the discontinuity occurs.
5. If the function comes to a Line Point or a Cusp Point, it is not differentiable at the exact x coordinate where the Point occurs.
6. If the function has an endpoint, it is not differentiable at the exact x coordinate of the endpoint.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Kinematics Problems

The instantaneous velocity is the derivative of the displacement function with respect to time and the instantaneous acceleration is the derivative of the velocity function with respect to time or the second derivative of the displacement function with respect to time. Some other questions that may be asked and how to solve them are listed below:

1. When is the particle at rest?
   Set the velocity function equal to zero $v(t) = 0$ and solve for t.
2. When or what is the maximum/minimum position or height?
   Set the velocity function equal to zero $v(t) = 0$ and solve for t. Plug this value t into the position function $s(t)$ to find the maximum or minimum position or height.
3. When or what is the maximum/minimum velocity or speed?
   Set the acceleration function equal to zero $a(t) = 0$ and solve for t. Plug this value t into the velocity function $v(t)$ to find the maximum or minimum velocity.
4. When is the particle moving in the positive/negative direction?
   Set the velocity function equal to zero $v(t) = 0$ and solve for t. Split the time domain $[0, \infty)$ with each of these time values t_1, t_2, t_3 found as endpoints to form separate intervals. Pick a test value t_{test} within each of these intervals, and plug this test value back into the velocity equation. The results will depend on the sign:
   If $v(t_{test}) > 0$ the particle will be moving in the positive direction for that entire interval containing t_{test}
   If $v(t_{test}) < 0$ the particle will be moving in the negative direction for that entire interval containing t_{test}
5. How much distance is traveled between $t = t_{initial}$ and $t = t_{final}$?
   Set the velocity function equal to zero $v(t) = 0$ and solve for t. These values t_1, t_2, t_3 found are endpoints of separate intervals. Evaluate the absolute value of the difference in the position for each interval:
   $$distance = |s(t_1) - s(t_{initial})| + |s(t_2) - s(t_1)| + |s(t_3) - s(t_2)| + \ldots + |s(t_{final}) - s(t_3)|$$

Implicit Differentiation

Implicit Differentiation finds the derivative of an equation that contains more than one variable in an Implicit form, a form where none of the variables are directly expressed in terms of the others. Taking the derivative of an Implicit equation results in the derivative of a variable with respect to a different variable. The steps for Implicit Differentiation:

1. Take the derivative both sides of the equation with respect to x, using the Derivative Identities. This includes the chain rule as every derivative of a y term should end up with a y' multiplying the same term.

2. Solve for y' by separating the terms that contain y'. Start by distributing any factors that may multiply a y' term. Collect all y' terms on one side of the equation by addition or subtraction. Factor out the y' from all terms on this side. Divide by the remaining expression. The Implicit Derivative will often be in terms of both x and y.

Equation of the Tangent Line or Normal Line

The equation of the tangent line or normal line to any point on the function graph can be found using the derivative:

1. If only the x coordinate is given in the problem, it will be necessary to plug it into the original equation to find the y coordinate to have an ordered pair point (x_1, y_1).
2. Find the derivative of the function and evaluate the derivative at the ordered pair point (x_1, y_1). Evaluating at the point yields the slope of the tangent line at that particular point
   For an Explicit Function $m_{tangent} = \frac{dy}{dx}(x_1)$ or $m_{tangent} = f''(x_1)$
   For an Implicit Function $m_{tangent} = \frac{dy}{dx}(x_1, y_1)$ or $m_{tangent} = f''(x_1, y_1)$
   The slope of the normal line will be the negative inverse of the slope of the tangent line $m_{normal} = -\frac{1}{m_{tangent}}$
3. Use the point-slope equation of a line $(y - y_1) = m\,(x - x_1)$ with the ordered pair point (x_1, y_1)
4. Solve the equation for y to put it into the slope-intercept equation form $y = m\,x + b$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Need Help?

Online Private Instruction
Anytime, Anywhere
Any High School!
Any College!

Individual or Small Group

Visit www.gradepeak.com

Call or Text 814.867.1777

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Related Rates
Related Rates calculate the rate at which one variable is changing given the rate at which other variables are changing:

1. Read the problem carefully. Draw a picture if applicable.
2. Introduce variables only to quantities that vary with time. Leave any constants as a number on the picture.
3. Write down the given values and rates in terms of the variables assigned.
4. Determine the Relation Equation, an equation that relates all time-dependent variables and constants in the problem. Geometry equations and the Pythagorean Theorem are often the Relation Equation:

Pythagorean Theorem $a^2 + b^2 = c^2$	Area of a circle $A = \pi r^2$	Volume of a sphere $V = \frac{4}{3}\pi r^3$
Area of a triangle $A = \frac{1}{2}bh$	Surface area of a sphere $S = 4\pi r^2$	Volume of a cylinder $V = \pi r^2 h$

5. To produce the Related Rates Equation, use Implicit Differentiation to find the derivative of the Relation Equation with respect to the time variable t. Each Chain Rule within the derivatives will produce a variable rate.
6. Substitute the given information into the Related Rates Equation to solve for the unknown value.

Differentials and Linearization
Differentials and Linearization are two related applications of the derivative and the slope of the tangent line.

Differentials
Differentials are the two terms that appear in the expression of the derivative $\frac{dy}{dx}$. The numerator dy is the y differential and the denominator dx is the x differential. The differentials represent an infinitely small change in the value of each of the variables. The two differentials can be separated by multiplication:

$$\frac{dy}{dx} = f'(x) \qquad\qquad dy = f'(x_{initial})dx$$

This set of equations is used to find the approximate change in the value of the function using the tangent line. The terms dx and dy represent infinitely small changes in x and y. Only $x_{initial}$ should be plugged into the derivative.

The next set of equations can be used to approximate one or both differentials as finite changes in the variables:

$$\Delta y \approx f'(x)\Delta x \qquad\qquad y(x_{final}) - y(x_{initial}) \approx f'(x_{initial})\,(x_{final} - x_{initial})$$

The second set of equations is used to find the approximate change in the value of the function. The terms Δx and Δy represent small changes in x and y. Only $x_{initial}$ should be plugged into the derivative. Some things to keep in mind:

1. Because it is the change in the x value, $dx = \Delta x$
2. Because dy is an infinitesimal change and Δy is a finite change: $\boldsymbol{dy \text{ is not necessarily equal to } \Delta y}$

The next set of equations can be used to find the actual change in the value of the function:

$$\Delta y = y_{final} - y_{initial} \qquad\qquad \Delta y = y(x_{final}) - y(x_{initial})$$

Linearization
Linearization can approximate the value of a function near a point through the equation of the tangent line at the point.

$$f(x) = f(a) + f'(a)\,(x - a)$$

1. If only the x coordinate a is given in the problem, it will be necessary to plug it into the original equation to find the y coordinate to have an ordered pair point $(x_1, y_1) = \big(a, f(a)\big)$.
2. Find the derivative of the function and evaluate the derivative at the ordered pair point $(x_1, y_1) = \big(a, f(a)\big)$. Evaluating at the point yields the slope of the tangent line at that particular point

   For an Explicit Function $\quad m_{tangent} = \frac{dy}{dx}(a) \quad$ or $\quad m_{tangent} = f'(a)$

   The slope of the normal line will be the negative inverse of the slope of the tangent line $m_{normal} = -\frac{1}{m_{tangent}}$
3. Use the point-slope equation of a line $(y - y_1) = m\,(x - x_1)$ with the ordered pair point $(x_1, y_1) = \big(a, f(a)\big)$
4. Solve the equation for y to put it into the slope-intercept equation form $y = m\,x + b$
5. Replace the variable y with the Linearization Function $L(x)$ to produce the linearization $L(x) = m\,x + b$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Economics Applications and the Derivative

Economics Functions

x is the number of units demanded or sold
$p(x)$ is the Demand Equation or the Price Equation, relating the price demanded to the number of units sold
$R(x) = x\, p(x)$ is the Revenue Function, relating the revenue generated to the number of units sold
$C(x)$ is the Cost Function, relating the cost generated to the number of units sold
$P(x) = R(x) - C(x)$ is the Profit Function, relating the profit generated to the number of units sold

Average Functions

$\overline{R(x)} = \frac{R(x)}{x}$ is the Average Revenue Function, or the average revenue generated per unit sold
$\overline{C(x)} = \frac{C(x)}{x}$ is the Average Cost Function, or the average cost generated per unit sold
$\overline{P(x)} = \frac{P(x)}{x}$ is the Average Profit Function, or the average profit generated per unit sold

Marginal Functions

$R'(x)$ is the Marginal Revenue Function, or the derivative of the Revenue Function
$C'(x)$ is the Marginal Cost Function, or the derivative of the Cost Function
$P'(x)$ is the Marginal Profit Function, or the derivative of the Profit Function

Marginal Average Functions

$\overline{R'(x)}$ is the Marginal Average Revenue Function, or the derivative of the Average Revenue Function
$\overline{C'(x)}$ is the Marginal Average Cost Function, or the derivative of the Average Cost Function
$\overline{P'(x)}$ is the Marginal Average Profit Function, or the derivative of the Average Revenue Function

Actual Production of the (x+1)st unit

$R(x+1) - R(x)$ is the Actual Revenue generated for just the (x+1)st unit
$C(x+1) - C(x)$ is the Actual Cost generated for just the (x+1)st unit
$P(x+1) - P(x)$ is the Actual Profit generated for just the (x+1)st unit

Approximate Production of the (x+1)st unit

$R'(x)$ is the Approximate Revenue generated for just the (x+1)st unit
$C'(x)$ is the Approximate Cost generated for just the (x+1)st unit
$P'(x)$ is the Approximate Profit generated for just the (x+1)st unit

Elasticity of Demand

The Elasticity of Demand is a function that relates a change in the Revenue Function with the change in the Unit Price.
$p(x)$ is the Price Equation or the Demand Equation, relating the price demanded to the number of units sold
$x(p)$ is the Price Equation or the Demand Equation, relating the number of units sold to the price demanded

$$E(p) = \frac{-p\, x'(p)}{x(p)}$$

1. If $E(p) > 1$ the Demand is Elastic; an increase in Unit Price will cause the Revenue to decrease and a decrease in Unit Price will cause the Revenue to increase
2. If $E(p) = 1$ the Demand is Unitary; an increase in Unit Price will cause the Revenue to stay the same and a decrease in Unit Price will cause the Revenue to stay the same
3. If $E(p) < 1$ the Demand is Inelastic; an increase in Unit Price will cause the Revenue to increase and a decrease in Unit Price will cause the Revenue to decrease

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Exponential Functions and Derivatives

An exponential function has a constant base a raised to a power that is a function of the independent variable x.

$$f(x) = a^{g(x)}$$

Definition of the Natural Base Number

The Natural Base e is the exact number when used as a base a of an exponential function e^x results in the graph having a tangent line slope or derivative with the exact value of 1 at the x coordinate $x = 0$ as seen by the derivative definition:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \qquad By\ defintion$$

The value of the Natural Base Number is irrational but can be found by definition to be $e = 2.1828 \ldots$

All other exponential functions with a different base a are then simply a scalar multiple of the Natural Base e exponential. The Natural Base e has several other relationships that result from its definition. These include:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \qquad\qquad e^{ab} = \lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^{bn}$$

Properties of Exponential Functions

All exponential functions with any base have the following properties

$a^x = a \cdot a \cdot a \cdot \ldots \cdot a \quad x\ many\ times$

$a^{x+y} = a^x\, a^y$

$a^{x-y} = \frac{a^x}{a^y}$

$a^{xy} = (a^x)^y$

$a^{\frac{x}{y}} = \sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$

$(ab)^x = a^x\, b^x$

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

$a^{-x} = \frac{1}{a^x}$

$a^0 = 1$

$a^1 = a$

$\lim_{x\to\infty} a^x = +\infty \quad if\ a > 1$

$\lim_{x\to-\infty} a^x = 0 \quad if\ a > 1$

$\lim_{x\to\infty} a^x = 0 \quad if\ 0 < a < 1$

$\lim_{x\to-\infty} a^x = +\infty \quad if\ 0 < a < 1$

The Limit of an Exponential Function

1. Directly evaluate the function at the limit by plugging in the value and using the results

   $\lim_{x\to\infty} a^x = +\infty \quad if\ a > 1 \qquad \lim_{x\to\infty} a^x = 0 \quad if\ 0 < a < 1$

   $\lim_{x\to-\infty} a^x = 0 \quad if\ a > 1 \qquad \lim_{x\to-\infty} a^x = +\infty \quad if\ 0 < a < 1$

   If the result is a finite number, that finite number is the value of the limit.

   If the result is $\frac{c}{0}$ or $\frac{\infty}{0}$ the value of the limit is $\pm\infty$

   If the result is $\frac{0}{0}, \frac{\infty}{\infty}$, or $\infty - \infty$ the value of the limit is indeterminate so continue to step 2.
2. Apply the exponential properties to simplify the equation and retry to directly evaluate the limit.
3. Continue applying other exponential properties until the limit is determinate.

The Derivative of an Exponential Function

The derivative of an Exponential Function is given by the following formulas:

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx} \qquad\qquad \frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

Where u is any function of x and $\frac{du}{dx}$ is the derivative of u to complete the chain rule within the derivative.

The first formula is the derivative of the Natural Base Exponential and the second is the derivative of any Exponential.

The Integral of an Exponential Function

The integral of an Exponential Function is given by the following formulas:

$$\int e^u\, du = e^u + C \qquad\qquad \int a^u\, du = \frac{a^u}{\ln a} + C$$

Where u is any function of x and du is the differential of u to complete the u substitution within the integral.

The first formula is the integral of the Natural Base Exponential and the second is the integral of any Exponential.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Logarithmic Functions and Derivatives

Logarithmic Functions of base a are the inverse of exponential functions of base a and are related by the following:

$$a^x = y \quad \leftrightarrow \quad \log_a y = x$$

The Natural Logarithmic Function and Change of Base Formula

The Logarithm with Natural Base e has unique properties and is called the Natural Logarithm. It has the special notation:

$$\log_e x = \ln x$$

A Logarithm of any base can be expressed as a Logarithm of the Natural Base e by the following Change of Base Formula:

$$\log_a x = \frac{\ln x}{\ln a}$$

Properties of Logarithmic Functions

All Logarithmic Functions with any base have the following properties

$a^x = y \quad \leftrightarrow \quad \log_a y = x$

$a^{\log_a x} = x$

$\log_a(x\, y) = \log_a x + \log_a y$

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(x^y) = y \, \log_a x$

$\log_a\left(\sqrt[y]{x}\right) = \frac{1}{y} \log_a x$

$\log_a(a^x) = x \, \log_a a = x$

$\log_a \frac{1}{x} = -\log_a x$

$\log_a 1 = 0$

$\log_a a = 1$

$\lim_{x\to\infty} \log_a x = +\infty \qquad if\ a > 1$

$\lim_{x\to 0^+} \log_a x = -\infty \qquad if\ a > 1$

$\lim_{x\to\infty} \log_a x = -\infty \qquad if\ 0 < a < 1$

$\lim_{x\to 0^+} \log_a x = +\infty \qquad if\ 0 < a < 1$

The Limit of a Logarithmic Function

1. Directly evaluate the function at the limit by plugging in the value and using the results

   $\lim_{x\to\infty} \log_a x = +\infty \qquad if\ a > 1 \qquad\qquad \lim_{x\to\infty} \log_a x = -\infty \qquad if\ 0 < a < 1$

   $\lim_{x\to 0^+} \log_a x = -\infty \qquad if\ a > 1 \qquad\qquad \lim_{x\to 0^+} \log_a x = +\infty \qquad if\ 0 < a < 1$

   If the result is a finite number, that finite number is the value of the limit.

   If the result is $\frac{c}{0}$ or $\frac{\infty}{0}$ the value of the limit is $\pm\infty$

   If the result is $\frac{0}{0}, \frac{\infty}{\infty}$, or $\infty - \infty$ the value of the limit is indeterminate so continue to step 2.
2. Apply the logarithmic or exponential properties to simplify the equation and retry to directly evaluate the limit.
3. Continue applying other logarithm or exponential properties until the limit is determinate.

The Derivative of a Logarithmic Function

The derivative of a Logarithmic Function is given by the following formulas:

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad\qquad \frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}$$

Where u is any function of x and $\frac{du}{dx}$ is the derivative of u to complete the chain rule within the derivative.

The first formula is the derivative of the Natural Base Logarithm and the second is the derivative of any Logarithm.

The Integral resulting in a Logarithmic Function

The integral resulting in a Logarithmic Function is given by the following formula:

$$\int \frac{1}{u}\, du = \ln u + C$$

Where u is any function of x and du is the differential of u to complete the u substitution within the integral.

Logarithmic Differentiation

Logarithmic Differentiation is a process through which to calculate the derivative of one of following functions:

The Power Exponential Function $\qquad y = f(x) = g(x)^{h(x)}$

A Multiple Complex Power Function $\qquad y = \frac{f(x)^b\, g(x)^c\, h(x)^d}{j(x)^p\, k(x)^q\, l(x)^r}$

1. Take the Natural Logarithm of both sides of the relation function $y = f(x)$. Simplify the $f(x)$ side using the Properties of Logarithmic Functions, especially watching for the following

   $\ln u^p = p \, \ln u \qquad \ln(u\, v) = \ln u + \ln v \qquad \ln\left(\frac{u}{v}\right) = \ln u - \ln v$
2. Differentiate both sides implicitly with respect to x, resulting in $\frac{1}{y}\frac{dy}{dx}$ on the y side of the relation function.
3. Solve the resulting equation for $\frac{dy}{dx}$ by multiplying both sides with the variable y.
4. Replace all y terms with the original relation function $y = f(x)$ for the explicit derivative in terms of x.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Compound Interest

For non-annual compounding, $A = P\left(1 + \frac{r}{m}\right)^{mt}$
A=Accumulated amount at the end of t years
P=Principal
r=nominal interest rate per year
m=number of conversion periods per year
t=term (number of years)

For continuous compounding, $A = Pe^{rt}$
A=Accumulated amount at the end of t years
P=Principal
r=annual interest rate compounded continuously
t=time in years

Rolles Theorem and Mean Value Theorem

Rolles Theorem and the Mean Value Theorem state a certain condition on the slope of a tangent line within a certain interval on which the function is both continuous and differentiable.

Rolles Theorem

- If the function f is continuous on the closed interval $[a, b]$
- If the function f is differentiable on the OPEN interval (a, b)
- If $f(a) = f(b)$
- **THEN** there is a number c in (a, b) such that $f'(c) = 0$

For a continous and differentiable function that on a particular interval happens to pass through the same y coordinate more than once, at some point within that interval the slope of the tangent line will have a value exactly equal to 0.

Mean Value Theorem

- If the function f is continuous on the closed interval $[a, b]$
- If the function f is differentiable on the OPEN interval (a, b)
- **THEN** there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

For a continous and differentiable function on a particular interval, at some point within that interval the slope of the tangent line will have a value exactly equal to the slope of the secant line between the two endpoints of the interval.

Graphing

The interesting points on a graph, extrema, plateaus, and inflection points can be found through the use of the calculus.

Function Domain

The Function Domain is the set or collection of all x values that can be plugged into a function. The Function Domain is All Real Numbers $(-\infty, +\infty)$ except in the following cases:

1. The function has a denominator, the denominator set as does not equal zero will give the Domain
   If $f(x) = \frac{g(x)}{h(x)}$ then the condition $h(x) \neq 0$ will give the Function Domain
2. The function has an even root, the inside part set greater than or equal to zero will give the Domain
   If $f(x) = \sqrt[n]{g(x)} = (g(x))^{\frac{1}{n}}$ with n even, then the condition $g(x) \geq 0$ will give the Function Domain

The Function Domain can be written in inequality notation, but it is more commonly written in bracket notation.
Square brackets [ ] are used for values that are on the end or extreme of an interval and are included in the interval.
Round brackets ( ) are used for values that are on the end or extreme of an interval but are not included in the interval.
Positive and Negative Infinity extreme values of an interval are never included in the interval and have round brackets.

Intercepts

Intercepts are the set of all points at which a function crosses the two coordinate axes. A function can have any number of x intercepts but can have at most only one y intercept. Some functions may not have an x intercept or y intercept.

x intercept

The set of all points at which the function crosses the x axis where the y coordinate is zero for the form $(x, 0)$.
To find the x intercepts, set the numerator function but not the denominator function equal to zero and solve for all x.
If $f(x) = \frac{g(x)}{h(x)}$ then the condition $g(x) = 0$ but $h(x) \neq 0$ will give the function x intercepts.

y intercept

The set of all points at which the function crosses the y axis where the x coordinate is zero for the form $(0, y)$.
To find the y intercept, evaluate the function at the x coordinate zero $y = f(0)$ and solve for the y.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Symmetry

Symmetry is the property at which a function shows identical behavior or an exact mirror image in its reflection around either the y axis alone known as y axis symmetry or around both the y axis and the x axis known as origin symmetry.

To test a function for either of the two types of symmetry:

1. Evaluate the function $f(x)$ at the negative x coordinate $f(-x)$

2. Simplify the function for all evaluations at the negative x coordinate following the rules:

   $(-x)^n = x^n \quad for\ n\ even$ $\qquad$ $(-x)^n = -x^n \quad for\ n\ odd$

3. Compare the simplified function $f(-x)$ to the original function $f(x)$.

   If $f(-x) = f(x)$ then the graph of the function will have y axis symmetry and display mirror image behavior between the first and second quadrants and mirror image behavior between the third and fourth quadrants.

   If $f(-x) = -f(x)$ then the graph of the function will have origin symmetry and display mirror image behavior between the first and third quadrants and mirror image behavior between the second and fourth quadrants.

Asymptotes and Limits at Infinity

Asymptotes are imaginary lines on a graph that the function approaches, but never reaches. There are three types:

1. **Vertical Asymptote**: Vertical lines along which a function approaches the extreme limits of either positive infinity or negative infinity. This will only occur in functions with a denominator. They are found by setting the denominator of the function equal to zero and solving for x for the equation of a vertical line. The function contains a denominator, and the limit involves the exact x coordinate that makes the expression in the denominator equal to zero. These limits result in either positive or negative infinity. For a Two Sided Limit, it is necessary to break it into two One Sided Limits. The type of infinity that each side produces is determined by plugging in the a and evaluating the overall sign of the expression.

   $$\lim_{x \to a} f(x) = \frac{c}{0} = \pm\infty \quad where\ c \neq 0$$

2. **Horizontal Asymptote**: Horizontal lines along which a function approaches at very large positive or very large negative x values of positive or negative infinity. These are found by calculating the limit of the function as x goes to both $+\infty$ and $-\infty$ then setting the limit equal to y for the equation of a horizontal line. The function contains a numerator and a denominator, and the limit involves the exact x coordinate that makes both the expression in the numerator and the expression in the denominator equal to infinity. The Function Limit will have a form of infinity divided by infinity, an Indeterminate Form that must be reworked with algebra.

   $$\lim_{x \to a} f(x) = \frac{\infty}{\infty} \qquad Indeterminate$$

   Divide the numerator and denominator by the highest order function in the denominator and retry the limit.

3. **Slant Asymptotes**: Slant lines along which a function approaches the extreme limits of either positive infinity or negative infinity. This will only occur in functions where the numerator and denominator are both power functions and the numerator is exactly one order higher than the denominator.

   $$f(x) = \frac{g(x)}{h(x)} \quad where\ g(x)\ is\ exactly\ one\ power\ higher\ than\ the\ h(x)$$

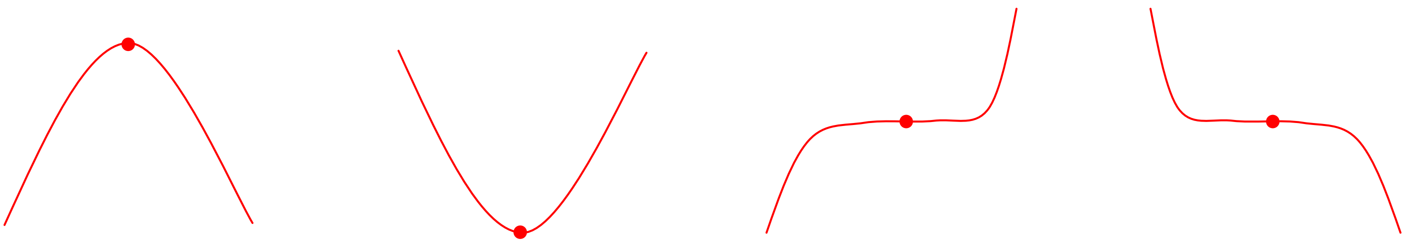
   The Slant Asymptote is found by long division of the numerator by the denominator, disregarding the remainder which will decrease to zero for large x, and then setting the result equal to y to get the equation of a slant line.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Critical Numbers

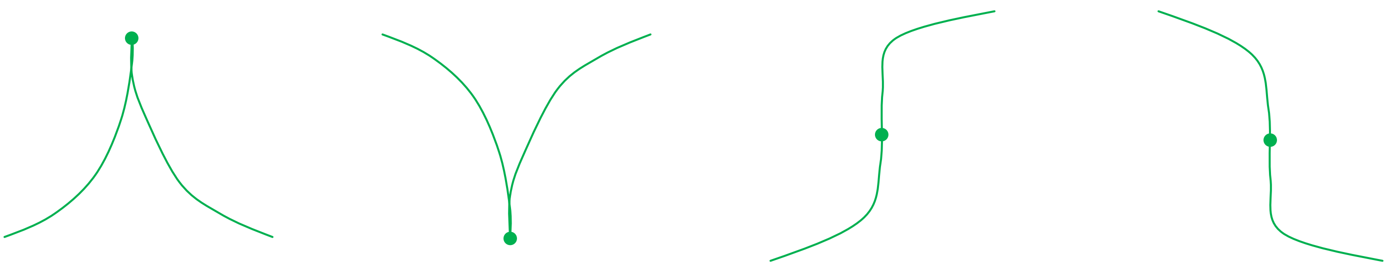
Critical Numbers are the x values where either the numerator or the denominator of both the first and second derivative of a function is equal to zero. **To be a critical number, the x values must be in the domain!** For the first derivative, the critical numbers are candidates for smooth extrema, cusp extrema, and plateaus. For the second derivative, the critical numbers are candidates for inflection points.

Critical Numbers of the First Derivative from the numerator will be one of the following depending on trend:



Increasing Decreasing | Decreasing Increasing | Increasing Increasing | Decreasing Decreasing

Critical Numbers of the First Derivative from the denominator will be one of the following depending on trend:



Increasing Decreasing | Decreasing Increasing | Increasing Increasing | Decreasing Decreasing

Critical Numbers of the Second Derivative from the numerator will be one of the following depending on concavity:

Concave Up Concave Down | Concave Down Concave Up | Concave Down Concave Up | Concave Up Concave Down

Critical Numbers of the Second Derivative from the denominator will be one of the following depending on concavity:

Concave Up Concave Up | Concave Down Concave Down | Concave Up Concave Down | Concave Down Concave Up

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Absolute Extrema

Absolute or Global Extrema are the points on a function graph that have the highest possible y coordinates. They will either be the critical points within the interval or at the endpoints of a given interval. To find them:

1. Always find the domain, unless given in the problem.
2. Find the first derivative. Separately set both the numerator and denominator equal to zero and solve for all x values. Only keep those in the domain. These are the critical numbers of the first derivative and are candidates to be absolute extrema.
3. Calculate and compare the y coordinates of all critical numbers and any endpoints of the domain. If two endpoints to the domain are not given or known, it may also be necessary to determine and horizontal asymptotes for comparison. The highest y coordinate achieved will be the absolute maximum point of the graph, and the lowest y coordinate achieved will be the absolute minimum point of the graph. It is possible to have more than one absolute maximum or minimum if there are extrema points with matching y coordinates.

First Derivative Test, Intervals of Trend, Relative Extrema, and Plateaus

First Derivative Test finds the Intervals of Trend, Relative Extrema, and Plateaus. Relative or Local Extrema are the points on a function graph that have the highest y coordinate compared to nearby point. The steps to the First Derivative Test:

1. Always find the domain, unless given in the problem.
2. Find the first derivative. Separately set both the numerator and denominator equal to zero and solve for all x values. Only keep those in the domain. These are critical numbers of the first derivative and are candidates to be either smooth extrema or plateaus if from the numerator or cusp extrema if from the denominator.
3. Create trend intervals by dividing all domain intervals with the critical numbers.
4. Make a chart with the following four columns: Interval, test value, sign of $f'(x)$, and trend. Pick a test value from each of the trend intervals. Plug each test value into a factored version of the first derivative to determine the sign of the first derivative. A positive first derivative at the test value means the function is increasing over the interval, and a negative first derivative at the test value means the function is decreasing over the interval.
   If the function is increasing to the left and decreasing to the right of a critical point, the point is a maximum
   If the function is decreasing to the left and increasing to the right of a critical point, the point is a minimum
   If the function is increasing on both sides or decreasing on both sides of the point, the point is a plateau

Second Derivative Test, Intervals of Concavity, and Inflection Points

Second Derivative Test finds the intervals of concavity and inflection points. The steps to the Second Derivative Test:

1. Always find the domain, unless given in the problem.
2. Find the second derivative. Separately set both the numerator and denominator equal to zero and solve for all x values. Only keep those in the domain. These are critical numbers of the second derivative and are candidates to be diagonal inflection points if from the numerator or vertical inflection points if form the denominator.
3. Create concavity intervals by dividing all domain intervals with the critical numbers.
4. Make a chart with the following four columns: Interval, test value, sign of $f''(x)$, and concavity. Pick a test value from each of the concavity intervals. Plug each test value into a factored version of the second derivative to determine the sign of the second derivative. A positive second derivative at the test value means the function is concave up over the interval, and a negative second derivative at the test value means the function is concave down over the interval. If the concavity changes over a critical number, the point is an inflection point.

Process for Graphing

1. Always find the domain, unless given in the problem.
2. Find all x intercepts and the y intercept if any exist.
3. Determine the symmetry of the graph if it exists.
4. Find vertical asymptotes, horizontal asymptotes, or slant asymptotes.
5. Use the First Derivative test to find trend intervals and determine the coordinates of minima, maxima, plateaus.
6. Use the Second Derivative Test to find concavity intervals and determine the coordinates of inflection points.
7. Draw the graph by connecting intercepts, extrema, and inflection points following symmetry and asymptotes.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Optimization

Optimization is a method for finding the best use of something by minimizing costs, materials, and time, or maximizing profit, area, volume, and productivity. These optimized values will occur at the appropriate absolute extrema of a function and can be found by comparing the critical points with the endpoints. The steps for optimization are:

1. Draw a picture, if applicable.
2. Define the variables and determine the known values and constants.
3. Write a function to relate the quantity to be optimized in terms of the variables and constants.
4. If the quantity to be optimized is in terms of two or more variables, find an equation that relates one variable in terms of the other and plug it into the quantity to be optimized function to form a function of just one variable
5. Find the first derivative of the one variable function to be optimized. Separately set both the numerator and denominator equal to zero and solve for all x values. Only keep those in the domain. These are the critical numbers of the first derivative and are candidates to be absolute extrema.
6. Calculate and compare the y coordinates of all critical numbers and any endpoints of the domain. The highest y coordinate achieved will be the absolute maximum point of the graph, and the lowest y coordinate achieved will be the absolute minimum point of the graph.
7. Be sure to READ and ANSWER the question! The values asked for can be determined from the absolute extrema.

Area Between Curves, Antiderivatives, and Integrals

The area between curves is the exact area of the region that exists between two or more curve graphs and often cannot be found through geometry alone. The antiderivative or integral is exactly the reverse operation of the derivative. For an antiderivative or integral the steps involved in the derivative are done in reverse operation and in reverse order. It can be shown that the definite antiderivative or definite integral and the area under the curve are fundamentally equal.

Riemann Sums

The Reimann Sum is a method to approximate the area between curves through the use of a collection of rectangles that approximately fit within the area itself. The rectangles are often set to consistently touch the curve at either their left endpoints (left corners), their right endpoints (right corners), or their midpoints (halfway between the left endpoints and right endpoints). As the number of rectangles increases the width of each becomes thinner and the approximation of the actual area becomes more precise though the calculations also increase. The formula for a Riemann Sum is:

$$Area \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \Delta x\left(f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)\right)$$

Where n is the number of rectangles, $f(x_i^*)$ is the height of each rectangle, and Δx is the width of each rectangle.

The width of each rectangle Δx over the interval for the area $[a, b]$ can be calculated as

$$\Delta x = \frac{b-a}{n}$$

The height of each rectangle $f(x_i^*)$ can be calculated by plugging each x coordinate x_i^* into the given function $f(x)$. The x coordinates can be found depending on the type of points to be used:

Left Endpoints
$x_i^* = a + (i-1)\,\Delta x$

Right Endpoints
$x_i^* = a + i\,\Delta x$

Midpoints
$$x_i^* = a + \frac{(2\,i-1)}{2}\,\Delta x$$

A summation limits as the number of rectangles n goes to infinity will equal the exact area or the exact definite integral.

$$Area = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Where it is often necessary to use the following identities:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n\,(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{2} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n\,(n+1)}{2}\right)^2$$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus has two parts that relates area between curves, antiderivatives, and integrals.

1. The first part relates the integral to the antiderivative, the reverse operation of the derivative in reverse order. If f is continuous on $[a,b]$, with a function $F(x)$ that is continuous on $[a,b]$, differentiable on (a,b) and defined by $F(x)=\int_{g(x)}^{h(x)} f(t)dt$ for $g(x)\le t\le h(x)$, then $F'(x)=f\big(h(x)\big)\,h'(x)-f\big(g(x)\big)\,g'(x)$

2. The second part relates the definite integral to the difference of the antiderivative evaluated at each bound. If f is continuous on $[a,b]$, then the definite integral defined by $\int_a^b f(x)dx=F(b)-F(a)$ is equal to the difference of the antiderivative evaluated at each bound, where F is any antidervative of f.

Indefinite Integral

The Indefinite Integral is exactly the antiderivative of a function. Since the derivative of a constant is zero, each Indefinite Integral will contain a constant C which itself may be positive, negative, or zero. Some Indefinite Integrals are:

$\int k dx = kx + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int \sin x\, dx = -\cos x + C$

$\int \cos x\, dx = \sin x + C$

$\int \sec^2 x\, dx = \tan x + C$

$\int \csc^2 x\, dx = -\cot x + C$

$\int \sec x \tan x\, dx = \sec x + C$

$\int \csc x \cot x\, dx = -\csc x + C$

The arithmetic properties of integrals are very similar to the arithmetic properties of derivatives

$$\int c\, f(x)\, dx = c\int f(x)\, dx \qquad \int [f(x)\pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Definite Integral

The Definite Integral is exactly equal to the area under a function curve graph on the interval $[a,b]$.

$$\int_a^b f(x)dx = F(x)|_a^b = F(b)-F(a)$$

The properties of definite integrals are

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \qquad \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Integration by u Substitution

Integration by u Substitution is used for integrals that contain products of functions that are related by one being the derivative of the other, or at least one being a constant multiple of the derivative of the other. This method is for:

Power Functions u^n Examples $x^2,\ x^7,\ \sqrt{x}=x^{\frac{1}{2}},\ \sqrt[3]{x}=x^{\frac{1}{3}},\ \frac{1}{x}=x^{-1},\ \frac{1}{x^3}=x^{-3},\ \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}},\ \frac{1}{\sqrt[3]{x}}=x^{-\frac{1}{3}}$

Trig Functions $\sin u,\ \cos u,\ \tan u$ Examples $\sin 2x,\ \cos 4x,\ \tan 3x$

Be sure to watch for a function and at least the variable part of its derivative together in the integral for u substitution.

How to Choose the u Function Correctly

The choice of u is determined by two different functions, one being the u itself and one being the variable part of the derivative du. With the following tips, you should be able to pick out the proper u for every integral encountered.

1. Trig Functions $\sin u,\ \cos u,\ \tan u$

If there is only one trig function, choose u to be the quantity inside of the trig function, but not the trig function itself. If there are two or more trig functions, choose u to be one of the trig functions such that the other is its derivative du.

2. Power Functions u^n where n is an integer or a fraction, and may be either positive or negative

Choose u to be the expression being raised to the power, especially if the expression is in the denominator.

The u Substitution Method

1. Choose the proper u following the rules such that the variable part of its derivative du is also present in the integral.
2. Differentiate the u to find du and divide or multiply any constants over with the du. The function that du and its constants equal to is the variable part of the du and will always contain the dx. This expression must be in the integral!
3. Replace expressions of x with u and the exact variable part of the derivative with du. Simplify this expression with algebra, and the result will always be one or more of the integrals from the Important Integral Formulas Table.
4. Integrate the functions of u, then replace every u in the integral result with the chosen u from step 1.

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.

Area Bounded Between Curves

The Area Bounded Between Curves is determined by a definite integral through the following steps:

1. Graph the function(s). This will determine the function orientations and which is top vs. bottom, or left vs. right.
2. Determine if it is vertically or horizontally simple. A vertically simple area will be bounded by only one curve on top and by only one curve on the bottom. It is not vertically simple if it is bounded both on the top and on the bottom by the same curve. A horizontally simple function will be bounded by only one curve on the right and by only one curve on the left. It is not horizontally simple if it is bounded both on the left and the right by the same curve.
3. A function that is neither vertically simple nor horizontally simple will need to be broken into intervals that are simple. The boundaries to the area will possibly change at each of the intersection points of the curves.
4. Calculate the area between the curves using the appropriate definite integral:

$$A = \int_a^b [f_{top}(x) - f_{bottom}(x)]dx \quad \text{vertically simple} \qquad A = \int_c^d [g_{right}(y) - g_{left}(y)]dy \quad \text{horizontally simple}$$

Average Value

Average Value $\overline{f(x)}$ of a curvilinear function graph can be determined exactly by integration

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Economics Applications of Integrals

Economics Applications of Integrals includes Consumer Surplus CS, Producer Surplus PS, Income Stream Accumulated Value A or Annuity Accumulated Value A and Income Stream Present Value PV or Annuity Present Value PV.

Income Stream

Income Stream $R(t)$ dollars per year deposited into or withdrawn from an account with interest rate r for time T years

$$\text{Accumulated Amount} \quad A = e^{rT}\int_0^T R(t)\,e^{-rt}\,dt \qquad \text{Present Value} \quad PV = \int_0^T R(t)\,e^{-rt}\,dt$$

Annuity

Annuity with m payments per year of size P dollars per payment from an account at interest rate r for time T years

$$\text{Accumulated Amount} \quad A = \frac{m\,P}{r}(e^{rT} - 1) \qquad \text{Present Value} \quad PV = \frac{m\,P}{r}(1 - e^{-rT})$$

Consumers Surplus

Consumers Surplus CS for a Demand Function $D(x)$,quantity sold $\bar{x}$, and unit market price $\bar{p}$ is the difference between what consumers would be willing to pay for $\bar{x}$ units of a commodity and what they actually pay for them.

$$CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\,\bar{x}$$

Producers Surplus

Producers Surplus PS for a Supply Function $S(x)$,quantity sold $\bar{x}$, and unit market price $\bar{p}$ is the difference between what producers actually receive for $\bar{x}$ units of a commodity and what they would be willing to receive for them.

$$PS = \bar{p}\,\bar{x} - \int_0^{\bar{x}} S(x)\,dx$$

© 2016 GradePeak, All Rights Reserved. Document Cannot Be Copied In Part Or Whole.